34 Friendship

From last time: if G is d-regular, girth 5, diameter 2, then $d = 2, 3, 7$, and maybe 57. These conditions are necessary, but do the graphs exist? Remember that the graphs must have $d^2 + 1$ vertices.

$d = 2$: C_5
$d = 3$: Petersen Graph
$d = 7$: Hoffman-Singleton graph
$d = 57$: unknown

Definition 34.1
A graph G that is **strongly regular**, denoted $\mathrm{SRG}(n, d, a, c)$, has n vertices and is d-regular such that

1. For any 2 adjacent vertices, they have a common neighbors.
2. For any 2 non-adjacent vertices, they have c common neighbors.

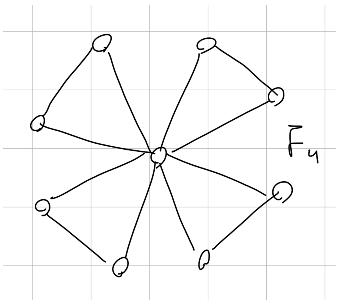
From earlier, a graph with girth 5, diameter 2, must have $a = 0$, because 2 adjacent vertices can never share a neighbor since the smallest cycle is 5.
If two non adjacent vertices share two common neighbors, then we have a 4-cycle, which can also not happen. So we must have $c = 1$.

So, we have classified all n and d for $\mathrm{SRG}(n, d, 0, 1)$.

Theorem 34.2 (Friendship)
At a party, suppose any 2 people have exactly 1 common friend.
Then the party contains a politician who is friends with everyone.

i.e. If G is a connected (finite) graph such that every 2 distinct vertices have 1 common neighbor, then there exists a vertex adjacent to all the others.

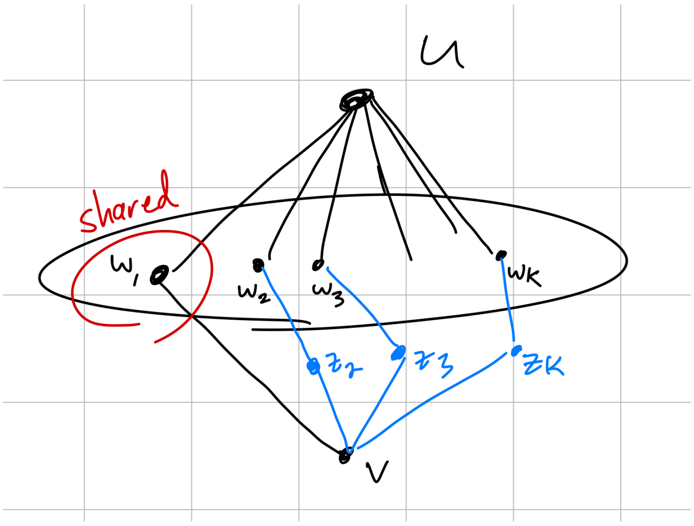
The friendship graph F_n on $2n + 1$ vertices: for F_4, this is a windmill graph.



Proof. Suppose the conclusion is false (no vertex is adjacent to all). With the "counterexample" graph, we show it must at least be a regular graph.

We then get a contradiction.
Let u NOT be adjacent to v, with $\deg(u) \geq \deg(v)$. Our goal is to show that $\deg(u) \leq \deg(v)$.
We then repeat this on all vertices so that they all have the same degree, which will make our graph regular.

Let $\deg(u) = k$. Recall that any 2 vertices share exactly one neighbor.



Since v and w_2 share a neighbor, we must have some z_2 which cannot be $w_3, w_4, \cdots, w_k$.
Otherwise, u and v would have 2 common neighbors.

So for each w_i, $i \geq 2$, there is a distinct z_i that they share.

However, v may be adjacent to other vertices. So, $\deg(v) \geq k = \deg(u)$.
If we repeat this process with all other vertices, all vertices must have the same degree.

Let a d-regular G be the counterexample. Suppose $d = 2$. This must be a cycle.
If the cycle is C_3, then this graph does satisfy the conclusion.
If the cycle is C_4 or larger, then it won't satisfy the given assumptions (any 2 vertices have exactly 1 common neighbor).
Thus, a counterexample would require $d \geq 3$.

Let A be the adjacency matrix of a counterexample.
Then, walks of length 2 $= A^2_{ij} = \begin{cases} d & i = j \\ 1 & i \neq j \end{cases}$
Then, $A^2 = J + (d-1)I$.
Let $\vec{v}$ be an eigenvector with eigenvalue $\lambda \neq d$ (so $\vec{v}$ is orthogonal to $\vec{1}$).

$$\implies A^2\vec{v} = \lambda^2\vec{v} = J\vec{v} + (d-1)\vec{v} = (d-1)\vec{v}$$

$$\implies \lambda^2\vec{v} = (d-1)\vec{v} \implies \lambda = \pm\sqrt{d-1}$$

Let $\lambda_1 = d$, $\lambda_2 = \sqrt{d-1}$ multiplicity m_2, and $\lambda_3 = -\sqrt{d-1}$ multiplicity m_3.

$\implies 0 = \mathrm{tr}(A) =$ sum of e-values $= d + m_2(\sqrt{d-1}) + m_3(-\sqrt{d-1})$.

Rearranging and squaring,

$$d^2 = (m_3 - m_2)^2(d-1)$$

If $m_3 = m_2 \implies d = 0$, fails.
If $m_3 \neq m_2$, then $d-1$ must divide d^2.. But $d-1$ also divides $d^2 - 1 = (d+1)(d-1)$.

Then $d-1$ divides 2 consecutive numbers. Then $d - 1 = 1 \implies d = 2$.

But we found earlier that a counterexample must have at least $d \geq 3$. Thus we have a contradiction. $\square$