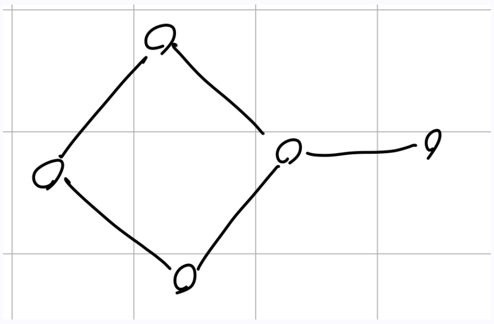
33 Eigenvalues of Graphs, Two Proofs (Section 5.3)

Example 33.1
Determine the spectrum and explain how you eliminated the wrong ones



1. $\{(-1)^{(2)}, 0, (1)^{(2)}\}$
2. $\{-3.2, -1.2, 0, 1.2, 3.2\}$
3. $\{-2.136, -0.662, 0, 0.662, 2.136\}$
4. $\{-2.1, (1)^{(2)}, 2, 2.1\}$

We know that the spectrum can not be 1 because the diameter is 3, and thus there must be at least 4 distinct eigenvalues.
It can not be 2 because $\Delta(G) = 3$, and $|\lambda| \leq \Delta(G)$.
We know the sum of eigenvalues must be 0, so it can not be 0.

Thus, the spectrum must be 3.

33.1 Two Proofs (Section 5.3)

Definition 33.2
The **girth** of a graph is the length of the shortest cycle.

Suppose a graph has n vertices. How many edges can we pack into the graph avoiding a 3-cycle?

Theorem 33.3
If G has n vertices and m edges and does not contain a 3 or 4 cycle (no C_3, C_4), then

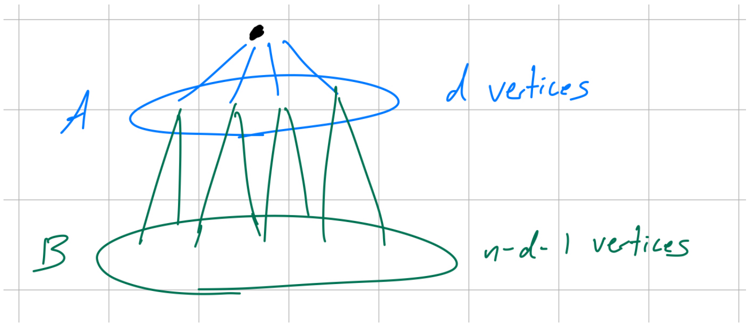
$$m \leq \frac{n\sqrt{n-1}}{2}$$

Moreover, equality occurs if G is d-regular, girth 5, and diameter 2 (everything can be reached in 2 steps).

What graphs obtain equality?

Lemma 33.4
If G is d-regular, girth 5, and diameter 2, then the total vertices that G has is $d^2 + 1$.

Proof. If we fix a single root point, notice that we can organize the graphs into sections A and B that look like the following:



Where nodes in set A are distance 1 away from the root, and nodes in set B are distance 2 away from the root.

Notice that there can not be any edges between nodes in A, because otherwise they would form a 3 cycle.
There are $d(d-1)$ edges from A to B: there are d nodes in A with degree d. There is 1 edge from the root to each one of these A vertices, and so each node in A must have $d-1$ edges to nodes in B.

As for edges from B to A, a vertex in B joints to exactly 1 vertex in A. Otherwise we would have a 4-cycle. So, there are $(n-d-1)\cdot(1)$ of these edges. So, we must have that

$$d(d-1) = (n-d-1)(1) \implies n = d^2+1$$

□

Theorem 33.5 (Hoffman-Singleton, 1960)
If G is a connected d-regular graph with girth 5 and diameter 2, then $d = 2, 3, 7$, and maybe 57.

Proof. We know $\lambda = d$ is an eigenvalue with e-vector $\vec{1} = (1, 1, \cdots 1)^T$.
Then, A^2_{ij} is the number of walks of length 2.

$$A^2_{ij} = \begin{cases} d & i = j \\ 0 & i \text{ adjacent to } j \\ 1 & i \text{ not adjacent to } j \end{cases}$$

$$\implies A^2 = dI + (J - A - I) \implies J = A^2 + A - (d-1)I$$

Let $\vec{u}$ be another eigenvector with eigenvalue λ. Note that $\vec{u}$ is orthogonal to $\vec{1}$ by the spectral theorem.

$$\implies \vec{0} = J\vec{u} = (A^2 + A - (d-1)I)\vec{u} = A^2\vec{u} + A\vec{u} - (d-1)\vec{u}$$

$$= \lambda^2\vec{u}^2 + \lambda\vec{u} - (d-1)\vec{u} = (\lambda^2 + \lambda - (d-1))\vec{u} = \vec{0}$$

$$\implies \lambda^2 + \lambda - (d-1) = 0$$

$$\implies \lambda_1 = \frac{-1+\sqrt{4d-3}}{2} \qquad \lambda_2 = \frac{-1-\sqrt{4d-3}}{2}$$

With multiplicity m_1 and m_2.

$$\implies m_1 + m_2 + 1 = \#\text{ vertices} = d^2 + 1 \text{ (by lemma)} \quad *$$

$$\implies \text{tr}(A) = \text{sum of e-values} = d + \lambda_1 m_1 + \lambda_2 m_2 = 0 \quad **$$

Multiply ** by 2 and adding to * to get

$$(m_1 - m_2)\sqrt{4d-3} = d^2 - 2d = \text{integer} \quad ***$$

If $m_1 = m_2$, then $d^2 - 2d = 0 \implies d = 0$ or $\boxed{2}$.

Otherwise if $m_1 \neq m_2$, then $\sqrt{4d-3} = k = \text{integer} \implies d = \frac{k^2+3}{4}$.
Sub into ***, and rearrange the terms,

$$16(m_1 - m_2)k - k^4 + 2k^2 = -15 \implies k(\qquad) = -15$$

So, k must divide 15, so $k = 1, 3, 5, 15$.
But $d = \frac{k^2+3}{4}$. At $k = 1, 3, 5, 15$, $d = 1, 3, 7, 57$. (but it is clear that 1 does not work). $\square$