32 Eigenvalues of Graphs

From last time, we had the Petersen graph, which has spectrum $\{3^{(1)}, (-2)^{(4)}, (1)^{(5)}\}$.
We know that K_5 can be decomposed into two copies of the cycle c_5.
What about K_{10} decomposing into 3 copies of the petersen graph?

Notice that in the K_5 case, we can see that we get

$$A(K_5) = A(C_5') + A(C_5'')$$

where C_5' and C_5'' are the two cycles it can decompose to.

Theorem 32.1
The edges of K_{10} can not be partitioned into 3 copies of the Petersen graph.

Proof. Assume K_{10} can be partitioned. Then

$$A(K_{10}) = A(P_1) + A(P_2) + A(P_3)$$

Let $\text{eig}_i(\lambda)$ be the eigenspace of λ for $A(P_i)$.
We know that $\vec{1}$ is an eigenvector, and all others are orthogonal (by spectral theorem).

We see that in the spectrum of the Petersen graph, $\lambda = 1$ has multiplicity 5.

$$\implies \dim(\text{eig}_1(\lambda = 1)) = \dim(\text{eig}_2(\lambda = 1)) = 5$$

K_{10} having 10 vertices means that the eigenvectors are all in $\mathbb{R}^{10}$.
Because we have $\vec{1}$ as an eigenvector, we only have 9 more linearly independent eigenvectors to make up $\mathbb{R}^{10}$, so $\text{eig}_1(1) \cap \text{eig}_2(1) \neq \varnothing$.

Let $\vec{x} \in \text{eig}_1(1) \cap \text{eig}_2(1)$, i.e. $A(P_1)\vec{x} = 1\vec{x}$ and $A(P_2)\vec{x} = 1\vec{x}$. Then, where J is the all ones matrix,

$$\implies A(K_{10})\vec{x} = (J - I)\vec{x} = J\vec{x} - I\vec{x} = \vec{0} - \vec{x} = -\vec{x}$$

Because $\vec{1} \cdot \vec{x} = 0$ because they are orthonormal.
Then,

$$\begin{aligned} -\vec{x} &= A(K_n)\vec{x} \\ &= (A(P_1) + A(P_2) + A(P_3))\vec{x} \\ &= A(P_1)\vec{x} + A(P_2)\vec{x} + A(P_3)\vec{x} \\ &= 1\vec{x} + 1\vec{x} + A(P_3)\vec{x} \\ A(P_3)\vec{x} &= -3\vec{x} \end{aligned}$$

But, -3 is not an eigenvalue of the Petersen graph, so this is a contradiction. □

Note 32.2
Recall that $\text{tr}(A)$ is the sum of the diagonal entries, which is the sum of the eigenvalues.

We have the following:

Theorem 32.3
The sum of the spectrum of any (simple) graph is 0.

Theorem 32.4 (Cayley-Hamilton)
For any square matrix A, if $p_A(z)$ is its characteristic polynomial, then $p_A(A) = 0$ matrix (A is a root of its characteristic polynomial).

Example 32.5

Take the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Then,

$$p_A(z) = (1-z)(2-z)^2 \implies p_A(A) = (I-A)(2I-A)^2 = 0 \text{ matrix}$$

Does there exist a lower degree polynomial $q(z)$ where $q(A) = 0$ matrix? Yes, there is $q(z) = (1-z)(2-z) \implies q(A) = (I-A)(2I-A) = 0$ matrix.

The lowest degree polynomial where this holds is the **minimal polynomial**.

Theorem 32.6

If A is diagonalizable (n linearly independent eigenvectors) (holds for adjacency matrix by spectral theorem) with distinct eigenvalues $\lambda_1, \cdots, \lambda_k$, then the minimal polynomial is $m(z) = (\lambda_1 - z)(\lambda_2 - z)\cdots(\lambda_k - z)$ (all linear factors).

Definition 32.7

The **diameter** of G is the greatest distance between any 2 vertices.

Theorem 32.8

If G has diameter d, then the spectrum has at least $d+1$ distinct eigenvalues, i.e. the diameter is strictly less than the number of distinct eigenvalues.

Proof. Let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues, and let A be the adjacency matrix.

We show that $I, A, A^2, \cdots, A^d$ is linearly independent.

Suppose $c_0 I + c_1 A + c_2 A^2 + \cdots + c_d A^d = 0$ matrix.
Recall that A^t represents the number of walks of length t between 2 vertices.

Then, diameter being d means there is some $A^d_{ij} \geq 1$, but $A^{d-1}ij = A^{d-2}_{ij} = \cdots = 0$.
Then, we must have $c_d = 0$ for $c_0 I + c_1 A + c_2 A^2 + \cdots + c_d A^d = 0$ to be true.

Similarly, there must be $A^{d-1}_{i_1 j_1} \geq 1$ with all lower powers equal to 0 (Otherwise, distance between i and j would not be d).
So, $c_d = 0$, and we continue until concluding that $c_0 = c_1 = \cdots = c_d = 0$.

Recall that the minimum polynomial $(\lambda_1 - z)(\lambda_2 - z)\cdots(\lambda_k - z)$ has degree k, and $c_0 I + c_1 A + c_2 A^2 + \cdots + c_d A^d = 0$ matrix.
Then, $I, A, \cdots, A^d$ are linearly independent.
Since they are linearly independent, there is no polynomial of degree d that results in $m(A) = 0$ matrix.

But the minimum degree polynomial has degree k, so $k > d$.
k is the number of distinct eigenvalues, which is $\geq d+1$. $\square$