31 Graph Spectrum (Section 5.2)

Last time, we had that the spectral theorem implies that if we have a square, real, symmetric matrix, then all eigenvalues are real, there are n orthonormal linearly independent eigenvectors, and the algebraic multiplicity equals the geometric multiplicity.

31.1 Graph Spectrum (Section 5.2)

Definition 31.1
A graph G is **d-regular** if every vertex has degree d, i.e. the total edges incident to each vertex is d.

Given the adjacency matrix of a d-regular graph, each row of the adjacency matrix should have d entries of 1s. We have the following:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & \cdots \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} = d \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$A\vec{v} = d\vec{v}$$

Thus for any d-regular graph, d is always an eigenvalue, with the corresponding eigenvector above.

Theorem 31.2
Let $\Delta(G)$ be the maximum degree of G. Then for any eigenvalue λ of the adjacency matrix A,

$$|\lambda| \leq \Delta(G)$$

In other words, the eigenvalue s never exceed the maximum degree of the graph, up to absolute value.

Moreover, equality occurs if and only if G is d-regular. In this case, $\lambda = d$ is always an eigenvalue with eigenvector

$$\vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

If G is also connected, $\lambda = d$ has multiplicity 1.

Proof. Let $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be an eigenvector with eigenvalue λ. Then if v_i is the largest entry of $\vec{v}$ up to absolute value,

$$[A] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$

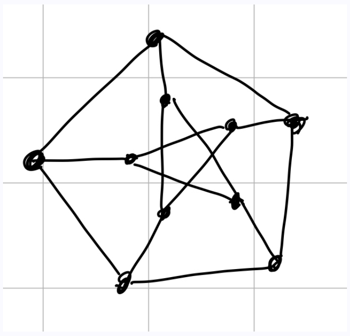
$$\implies |\lambda||v_i| = |\lambda v_i| = \left| \sum_{j=1}^{n} a_{ij} v_j \right| \leq \sum_{j=1}^{n} a_{ij}|v_j| \leq \sum_{j=1}^{n} a_{ij}|v_i| = |v_i| \cdot \text{degree of vertex } i \leq |v_i|\Delta(G)$$

$$\implies |\lambda||v_i| \leq |v_i|\Delta(G) \implies |\lambda| \leq \Delta(G)$$

Where the first inequality is derived from the triangle inequality. ☐

Definition 31.3
The **spectrum** of G is the **multiset** of eigenvalues of the adjacency matrix of G.

Example 31.4
The following graph is called the Petersen Graph:



It has 10 vertices, 15 edges, is 3-regular, and the smallest cycle is length 5.

It is often used as counterexamples in graph theory because of its strange properties.

By computer, the spectrum of the graph is

$$\{(3)^{(1)}, (1)^{(5)}, (-2)^{(4)}\}$$

Where the superscripts represent the multiplicity of the eigenvalue. If multiplicity is 1, the superscript can be omitted.

Example 31.5
The **complete graph** K_n is the graph where every 2 vertices has an edge between them. Notice that K_n is a $n-1$ regular graph.

Theorem 31.6
The spectrum of K_n is

$$\{(n-1)^{(1)}, (-1)^{(n-1)}\}$$

Proof. Let J be the all ones matrix (every entry is 1). We know that $\lambda = n-1$ has eigenvector $\vec{1}$.
Let $\vec{v}$ be another eigenvector not associated with $\lambda = n-1$.

$$A(K_n)\vec{v} = (J-I)\vec{v} = J\vec{v} - I\vec{v} = J\vec{v} - \vec{v}$$

Here, in $J\vec{v}$, notice we are dotting $\vec{1} \cdot \vec{v}$. Because these are both eigenvectors, by the spectral theorem, they must be orthogonal, so $\vec{1} \cdot \vec{v} = 0$.

$$J\vec{v} - \vec{v} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \vec{v} = -\vec{v}$$

$$\implies A(K_n)\vec{v} = -\vec{v}$$

So all other eigenvalues are -1. □

Given the complete graph K_5, can we partition the edges into two copies of C_5, the cycle of length 5?

By drawing K_5 as a pentagon with a star inscribed inside, it is simple to see that we can partition the edges to create two 5-cycles.

What about the edges of K_{10}? Can they be decomposed into 3 copies of the Petersen Graph?