30 Polynomial Spaces, Basics of Graphs (Section 5.1)

30.1 Polynomial Spaces

From last time, we had:

$$f_{\vec{w}_i}(\vec{x}) = \left(||\vec{x} - \vec{w}_i||^2 - d_1^2\right)\left(||\vec{x} - \vec{w}_i||^2 - d_2^2\right)$$

We showed that functions must be linearly independent, and they lie in

$$\text{span}\left(\{1, x_i, x_ix_j, x_i^2, x_i \sum x_k^2, \left(\sum_{i=1}^{n} x_i^2\right)^2\}\right)$$

Thus the functions lie in a space spanned by

$$1 + n + \binom{n}{2} + n + n + 1 = \frac{(n+1)(n+4)}{2}$$

Since the functions $f_{\vec{w}_i}(\vec{x})$ must be linearly independent, they cannot exceed the dimension of the space, and remember that the number of polyonmials is equal to the number of points in the space. # polynomials = # points $\leq$ dimension of the space = $\frac{(n+1)(n+4)}{2}$.

Note that equality can not be obtained (we can not reach this upper bound).

30.2 The Basics of Graphs (Section 5.1)

Recall the **characteristic polynomial** of A is $p_A(\lambda) = \det(A - \lambda I)$. The roots are the eigenvalues of A i.e. λ such that $A\vec{v} = \lambda\vec{v}$, where $\vec{v} \neq \vec{0}$ is an **eigenvector**.

The **algebraic multiplicity** of λ is how many times it occurs in $p_A(\lambda)$.

The **geometric multiplicity** of λ is the number of linearly independent eigenvectors associated with λ, i.e. the dimension of the eigenspace of λ.

Example 30.1

$A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \implies p_A(\lambda) = (4 - \lambda)^2$

So $\lambda = 4$ is an eigenvalue with algebraic multiplicity 2.

But we find the eigenvectors for $\lambda = 4$ are $\begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1,\ x_1 \in \mathbb{R} \setminus \{0\}$. There is only one linearly indepedent eigenvector, so the geometric multiplicity of $\lambda = 4$ is only 1.

Properties of Eigenvalues:

1. Trace of A, $\text{tr}(A)$ = sum of the diagonal entries = sum of all eigenvalues.
2. $\det(A)$ = product of eigenvalues.

Theorem 30.2 (Spectral Theorem)

Let A be an $n \times n$, real, symmetric matrix (i.e. $A = A^T$). Then,

1. All eigenvalues are real
2. For any eigenvalue λ, the algebraic multiplicity is equal to the geometric multiplicity
3. Matrix A has an orthonormal basis of eigenvectors that spans $\mathbb{R}^n$.

Definition 30.3
A graph is **simple** if there are no multiple edges or loops.

The **adjacency matrix** of a (simple) graph G on n vertices is

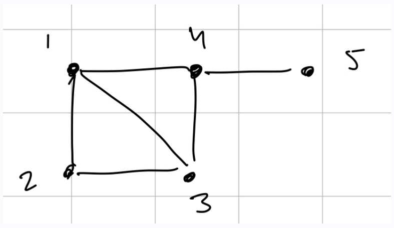
$$(A(G))_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{otherwise} \end{cases}$$

So, if $i \sim j$ form an edge, then i is adjacent to j and j is adjacent to i, which means that $A_{ij} = A_{ji} = 1$. So, the adjacency matrix must be symmetric.

Thus, the adjacency matrix satisfies the assumptions of the spectral theorem! This is the basis of spectral graph theory.

Theorem 30.4
If $A = A(G)$ is the adjacency matrix, then the ijth entry of A^k is the total walks of length k that go from vertex i to j.

Example 30.5
Suppose we have the following graph:



Then we have the following adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\implies A^4 = \begin{bmatrix} 16 & 9 & 15 & 10 & 6 \\ 9 & 10 & 9 & 12 & 2 \\ 15 & 9 & 16 & 10 & 6 \\ 10 & 12 & 10 & 15 & 2 \\ 6 & 2 & 6 & 2 & 3 \end{bmatrix}$$

Here, entry $(2, 5)$ is 2. So, the total number of walks of length 4 from 2 to 5 is 2.
These walks are $2 \to 3 \to 1 \to 4 \to 5$, and $2 \to 1 \to 3 \to 4 \to 5$.

Entry $(5, 5)$ is 3 (number of walks of length 4 that start and end at 5). These are:
$5 \to 4 \to 5 \to 4 \to 5$
$5 \to 4 \to 1 \to 4 \to 5$
$5 \to 4 \to 3 \to 4 \to 5$

Given a graph, we may label the vertices differently. How does this effect the eigenvalues of the two matrices? Surprisingly, the eigenvalues do not change.

Theorem 30.6
Given graph G, the eigenvalues are determined: Any relabeling of the vertices does not change the eigenvalues.