3 Affine Combinations (Section 2.1)

3.1 Introduction

Definition 3.1

Let $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$, $c_1, \cdots, c_m \in \mathbb{R}$. An **affine combination** of $\vec{v}_1, \cdots, \vec{v}_m$ is a linear combination $\sum_{i=1}^m c_i\vec{v}_i$ such that $\sum_{i=1}^m c_i = 1$.

The set of all affine combinations of a set of vectors S is called the **affine hull**, denoted $\text{aff}(S)$.
This is similar to how the span of a set is all linear combinations of a set. An affine hull is all affine combinations of a set.

Example 3.2

1. $S = \{\vec{v}_1\} \implies \text{aff}\, S = \{\vec{v}_1\}$
   Imagine in previous linear algebra courses, how for $T = \{\vec{v}_1\}$, $\text{span}(T) = \{c\vec{v}_1 : c \in \mathbb{R}\}$
2. $S = \{\vec{v}_1, \vec{v}_2\} \implies \text{aff}(S)$ has vectors of the form $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2$, where $c_1 + c_2 = 1$ or $c_1 = 1 - c_2$.
   So we can rewrite $\vec{y}$ as $\vec{y} = (1 - c_2)\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_1 + c_2(\vec{v}_2 - \vec{v}_1)$, which is equivalent of the same form as $\vec{p}_0 + t\vec{v}$, which is the parameterization of a line from calculus.
   Thus, $\text{aff}(S)$ is a line through $\vec{v}_1$ in the direction of $\vec{v}_2 - \vec{v}_1$ (assuming the vectors are not multiples of each other).

   In linear algebra, we say two vectors are linearly independent if their span creates a plane. The analogue here is that the affine hull of two vectors just creates a line - it reduces the dimensionality down by one dimension because of the restriction on the coefficients.

   Similar, one can show if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not all on a line, $\text{aff}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ is a plane.

How do we determine if $\vec{y} \in \text{aff}(S)$?

Theorem 3.3

Vector $\vec{y} \in \mathbb{R}^n$ is an affine combination of $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$ if and only if $\vec{y} - \vec{v}_1$ (or use any of $\vec{v}_2, \cdots, \vec{v}_m$) is a linear combination of $\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_1, \cdots, \vec{v}_m - \vec{v}_1$.

Proof. ($\implies$): Given that $\vec{y}$ is an affine combination, $\vec{y} = \sum_{i=1}^m c_i\vec{v}_i$, $\sum_{i=1}^m c_i = 1$.
Then $\vec{y} - \vec{v}_1 = (c_1 - 1)\vec{v}_1 + \sum_{i=2}^m c_i\vec{v}_i$ (note that $c_1 - 1 = -c_2 - c_3 - \cdots - c_m$)
Thus, we have that

$$\begin{aligned}\vec{y} - \vec{v}_1 &= (-c_2 - c_3 - \cdots - c_m)\vec{v}_1 + \sum_{i=2}^m c_i\vec{v}_i \\ &= c_2(\vec{v}_2 - \vec{v}_1) + c_3(\vec{v}_3 - \vec{v}_1) + \cdots + c_m(\vec{v}_m - \vec{v}_1)\end{aligned}$$

Which means that $\vec{y} - \vec{v}_1$ is a linear combination of $\vec{v}_2 - \vec{v}_1, \cdots, \vec{v}_m - \vec{v}_1$ □

Proof. ($\impliedby$) Given that $\vec{y} - \vec{v}_1$ is a linear combination of $\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_1, \cdots, \vec{v}_m - \vec{v}_1$, we have that

$$\begin{aligned}\vec{y} - \vec{v}_1 &= k_2(\vec{v}_2 - \vec{v}_1) + k_3(\vec{v}_3 - \vec{v}_1) + \cdots + k_m(\vec{v}_m - \vec{v}_1) \\ \vec{y} &= (1 - k_2 - k_3 - \cdots - k_m)\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + \cdots + k_m\vec{v}_m\end{aligned}$$

Here, we can see that $\vec{y}$ is a linear combination of $\vec{v}_1, \cdots, \vec{v}_m$, and the coefficients of $\vec{v}_1, \cdots, \vec{v}_m$ add up to 1.
Thus, $\vec{y}$ is an affine combination of $\vec{v}_1, \cdots, \vec{v}_m$. □

Example 3.4
Express $\vec{y} = (17, 1, 5)$ as an affine combination of $\{(-3, 1, 1), (0, 4, -2), (4, -2, 6)\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

The first step we wish to take is to represent $\vec{y} - \vec{v}_1$ as a linear combination of $\vec{v}_2 - \vec{v}_1$ and $\vec{v}_3 - \vec{v}_1$:

$$\vec{y} - \vec{v}_1 = (20, 0, 4),\ \vec{v}_2 - \vec{v}_1 = (3, 3, -3),\ \vec{v}_3 - \vec{v}_1 = (7, -3, 5).$$

We row reduce $\left[\begin{array}{cc|c} 3 & 7 & 20 \\ 3 & -3 & 0 \\ -3 & 5 & 4 \end{array}\right]$ down to $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$, which implies that

$$\vec{y} - \vec{v}_1 = 2(\vec{v}_2 - \vec{v}_1) + 2(\vec{v}_3 - \vec{v}_1)$$

Now, we just have to solve for $\vec{y}$:

$$\vec{y} = \vec{v}_1 + 2(\vec{v}_2 - \vec{v}_1) + 2(\vec{v}_3 - \vec{v}_1) = -3\vec{v}_1 + 2\vec{v}_2 + 2\vec{v}_3$$

Here, the sum of the coefficients here is 1, so we know this is an affine combination.

Note 3.5
If $\vec{y} \in \mathbb{R}^n$, and $\vec{v}_1, \cdots, \vec{v}_n \in \mathbb{R}^n$ is a basis, to check if $\vec{y}$ is in the affine hull, simply find the unique linear combination. Then, we just have to check if this linear combination has coefficients that sum to 1 (the unique linear combination is the only way you can write the combination anyways, so we only have to check if this one is affine).

Definition 3.6
A set S is **affine** if for any $\vec{p}, \vec{q} \in S$, $(1-t)\vec{p} + t\vec{q} \in S$ for any $t \in \mathbb{R}$.

Definition 3.7
A **translate** of $S \subseteq \mathbb{R}^n$ by $\vec{p} \in \mathbb{R}^n$ is $S + \vec{p} = \{\vec{s} + \vec{p} : \vec{s} \in S\}$ (similar to cosets in abstract algebra).
A **flat** is a translate of a subspace of $\mathbb{R}^n$. (Imagine a plane that is shifted to not pass the origin)

Theorem 3.8
We have that a set S is affine if and only if S is a flat.

Definition 3.9
The (standard) **homogeneous form** of $\vec{v} \in \mathbb{R}^n$ is

$$\tilde{v} = \begin{bmatrix} \vec{v} \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$$