29 Polynomial Spaces (Section 4.2)

29.1 Polynomial Spaces (Section 4.2)

Before: the maximum number of points in $\mathbb{R}^n$ with all the same distance? We showed that $m \leq n$.

Now: the maxmimum number of points where the distance can be two values between any 2 points.
4 points on a square work: any two pairwise points are distance 1 or $\sqrt{2}$.

We look at the vector space of polynomials in n variables with real coefficients, denoted $\mathbb{R}[x_1, x_2, \cdots, x_n]$. Note that there is no restriction on the degree of the polynomials, so this space has infinite dimensions.

Example 29.1
$f(\vec{x}) = f(x_1, x_2, x_3, x_4) = x_1^2 - 4x_1x_3 - x_2^3 + x_4 \in \mathbb{R}[x_1, x_2, x_3, x_4]$

Theorem 29.2
Let $f_1(\vec{x}), \cdots, f_k(\vec{x}) \in \mathbb{R}[x_1, \cdots, x_n]$.

Assume there exists points $\vec{v}_1, \cdots, \vec{v}_k \in \mathbb{R}^n$ such that

$$f_i(\vec{v}_j) = \begin{cases} 0 & \text{if } i \neq j \\ \text{nonzero} & \text{if } i = j \end{cases}$$

Then $f_1(\vec{x}), f_2(\vec{x}), \cdots, f_k(\vec{x})$ is linearly independent in $\mathbb{R}[x_1, x_2, \cdots, x_n]$.

Example 29.3
$f_1(x_1, x_2, x_3) = x_1 - x_2 + x_3 \in \mathbb{R}[x_1, x_2, x_3]$
$f_2(x_1, x_2, x_3) = x_1^5 - 100x_3$
$f_3(x_1, x_2, x_3) = x_2$

We are trying to check if the three functions are linearly independent:

$$c_1(x_1 - x_2 + x_3) + c_2(x_1^5 - 100x_3) + c_3(x_2) = 0$$

Intuitively, we can see that because f_2 has a 5th degree term, there is no way we can cancel it out with the other functions, so we can't have linear dependence.

Let

$$\begin{aligned} \vec{v}_1 &= (1, 0, 1/100) \\ \vec{v}_2 &= (1, 0, -1) \\ \vec{v}_3 &= (1, 101/100, 1/100) \end{aligned}$$

We can see that

$$\begin{aligned} f_1(\vec{v}_1) &= f_1(1, 0, 1/100) \neq 0 \\ f_1(\vec{v}_2) &= f_1(1, 0, -1) = 0 \\ f_1(\vec{v}_3) &= f_1(1, 101/100, 1/100) = 0 \end{aligned}$$

Similarly, $f_2(\vec{v}_1) = f_2(\vec{v}_3) = 0$, but $f_2(\vec{v}_2) \neq 0$. And $f_3(\vec{v}_1) = f_3(\vec{v}_2) = 0$ but $f_3(\vec{v}_3) \neq 0$.

Thus, by the theorem, $f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x})$ is linearly independent.

Proof. (Of the above theorem)
Suppose

$$c_1 f_1(\vec{x}) + c_2 f_2(\vec{x}) + \cdots + c_k f_k(\vec{x}) = 0$$

We will show that the c_i's must be 0.

Plug in $\vec{v}_1 \implies$

$$c_1 f_1(\vec{v}_1) + c_2 f_2(\vec{v}_1) + \cdots + c_k f_k(\vec{v}_1) = 0$$

Notice that all $f_i(\vec{v}_1) = 0$ for $i \neq 1$. We also know that $f_1(\vec{v}_1)$ is nonzero by our assumptions.

$$\implies c_1 f_1(\vec{v}_1) = 0 \implies c_1 = 0$$

Continue and plug in $\vec{v}_i$, $2 \leq i \leq k$, which forces $c_2 = c_3 = \cdots = c_k = 0$.
This implies that $f_1(\vec{x}), \cdots, f_k(\vec{x})$. □

Theorem 29.4
The maximum points in $\mathbb{R}^n$ such that the pairwise distances between any two points takes on values d_1 or d_2 is at most $\frac{1}{2}(n+1)(n+4)$.

Proof. (Sketch)
Let $\vec{w}_1, \cdots, \vec{w}_m$ be the points satisfying the condition, i.e. $||\vec{w}_i - \vec{w}_j|| = d_1$ or d_2.
Represent each $\vec{w}_i$ by a polynomial $f_{\vec{w}_i}(\vec{x})$. We will show with the condition of the distances, that these corresponding polynomials must be linearly independent.
We need to see what (finite) dimensional subspace these polynomials lie in.

With linear independence, we then know that the number of polynomials = the number of points with pairwise distances d_1 or $d_2 \leq$ dimension of the subspace they lie in.

What are the polynomials? Let

$$f_{\vec{w}_1}(\vec{x}) = \left(||\vec{x} - \vec{w}_1||^2 - d_1^2\right)\left(||\vec{x} - \vec{w}_1||^2 - d_2^2\right)$$

$$f_{\vec{w}_2}(\vec{x}) = \left(||\vec{x} - \vec{w}_2||^2 - d_1^2\right)\left(||\vec{x} - \vec{w}_2||^2 - d_2^2\right)$$

$$\vdots$$

$$f_{\vec{w}_m}(\vec{x}) = \left(||\vec{x} - \vec{w}_m||^2 - d_1^2\right)\left(||\vec{x} - \vec{w}_m||^2 - d_2^2\right)$$

Plug in $\vec{w}_1$:

$$f_{\vec{w}_1}(\vec{w}_1) = (0 - d_1^2)(0 - d_2^2) = d_1^2 d_2^2 \neq 0$$

If we plug in any other $\vec{w}_i$ with $i \neq 1$:

$$f_{\vec{w}_1}(\vec{w}_i) = (||\vec{w}_i - \vec{w}_1||^2 - d_1^2)(||\vec{w}_i - \vec{w}_1||^2 - d_2^2) = 0$$

Since either $||\vec{w}_i - \vec{w}_1|| = d_1$ or d_2.
We find that $f_{\vec{w}_i}(\vec{w}_j) = 0$ if $i \neq j$, but $f_{\vec{w}_i}(\vec{w}_j) \neq 0$ if $i = j$.

This is the condition of the earlier theorem! So $f_{\vec{w}_1}, \cdots, f_{\vec{2}_m}$ must be linearly independent.

How many polynomials do you need?
We need, based on the example given in class,

$$\{1, x_i, x_i x_j, x_i^2, x_j \sum_{k=1}^{n} x_k^2, \left(\sum_{k=1}^{n} x_k^2\right)^2 \}$$

The number of terms of each "type" are 1, n, $\binom{n}{2}$, n, n, and 1 respectively.
Adding up these totals together, we get $\frac{1}{2}(n+1)(n+4)$. □