28 Oddtown and Eventown, Other Results

From last time, we had that if $A_1, \cdots, A_m \subseteq [n]$, $|A_i|$ even, $|A_i \cap A_j|$ even, then $m \leq 2^{\lfloor n/2 \rfloor}$.

Proof. Let $s = \{\vec{v}_1, \cdots, \vec{v}_m\}$ be characteristic vectors $W = \text{span}(S)$.

We showed that $S \subseteq S^\perp \implies W \subseteq W^\perp$.

We know that the number of sets is $m = |S| \leq |\text{span}(S)| = |W|$, which means the number of sets will have an upper bound if we know $|W|$.

It can be shown if W is a subspace of a vector space of dimension n,

$$\dim(W) + \dim(W^\perp) = n$$

Note that here we are in $\mathbb{F}_2^n$.
But we know that $W \subseteq W^\perp$.
So, we have that $\dim(W) \leq \lfloor \frac{n}{2} \rfloor$.

Suppose W has a basis $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_{\lfloor n/2 \rfloor}$.

How large is W? We know that any vector $\vec{z} \in W$ is of the form $\vec{z} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_{\lfloor n/2 \rfloor}\vec{u}_{\lfloor n/2 \rfloor}$, with $a_i \in \mathbb{F}_2$.

Thus, there can be at most $2^{\lfloor n/2 \rfloor}$ such vectors in W.
So, the number of sets $= m = |S| \leq |\text{span}(S)| = |W| \leq 2^{\lfloor n/2 \rfloor}$. □

28.1 Other Cases

Theorem 28.1
Let $A_1, \cdots, A_m \subseteq [n]$ such that

1. $|A_i|$ is odd for $1 \leq i \leq m$
2. $|A_i \cap A_j|$ is odd for $1 \leq i \neq j \leq m$

Then $m \leq 2^{\lfloor (n-1)/2 \rfloor}$.

Theorem 28.2 (Reverse Oddtown)
Let $A_1, \cdots, A_m \subseteq [n]$, such that

1. $|A_i|$ is even, $1 \leq i \leq m$
2. $|A_i \cap A_j|$ is odd, $1 \leq i \neq j \leq m$

Then if n is odd, $m \leq n$.
If n is even, $m \leq n - 1$.

Theorem 28.3 (Fisher's Inequality)
Fix an integer k, $1 \leq k \leq n$.
Let $A_1, \cdots, A_m \subseteq [n]$ be distinct such that

$$|A_i \cap A_j| = k \qquad 1 \leq i \neq j \leq m$$

Then $m \leq n$.

Proof. Let $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$ be the usual characteristic vectors.
It suffices to show the set of vectors is linearly independent over $\mathbb{R}^n$.

Recall that $\vec{v}_i \cdot \vec{v}_j = |A_i \cap A_j| = k$, $\vec{v}_i \cdot \vec{v}_i = |A_i|$.

Let $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$. (We will show that all a_i's must be 0)

$$\begin{aligned}
0 = \vec{u}\cdot\vec{u} &= (a_1\vec{v}_1 + \cdots + a_m\vec{v}_m)\cdot(a_1\vec{v}_1 + \cdots + a_m\vec{v}_m) \\
&= \sum_{i=1}^{m} a_i^2(\vec{v}_i\cdot\vec{v}_i) + 2\sum_{1\le i<j\le m} a_i a_j(\vec{v}_i\cdot\vec{v}_j) \\
&= \sum_{i=1}^{m} a_i^2|A_i| + 2\sum_{1\le i<j\le m} a_i a_j k \\
&= \sum_{i=1}^{m} a_i^2(|A_i| - k) + k\left(\sum_{i=1}^{m} a_i\right)^2 = 0
\end{aligned}$$

Note that there is at most 1 set of size k. (all must be larger).
Suppose first that no set has size k. Then, $|A_i| - k > 0$, so $a_i^2(|A_i| - k)$ will be strictly positive unless all $a_i's$ are 0.

Otherwise, if say $|A_1| = k$, we also fine that $a_1 = a_2 = \cdots = a_m = 0$ necessarily.

Thus, $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is necessarily linearly independent over $\mathbb{R}^n$. The total number of vectors can not exceed the dimension of the space, so $m \le n = \dim(\mathbb{R}^n)$.

□

Example 28.4
Given m points on the xy-plane, not all on 1 line, show the pairs of points always define at least m distinct lines.

Solution: Let L be the set of lines created by the points. We want $|L| \ge m$.
Let $A_i = \{\text{lines } l \subseteq L \text{ that contain point } p_i\}$.

$A_i \cap A_j$ represents all lines that contain p_i and p_j.

We know that $|A_i \cap A_j| = 1$. Thus, we have that $A_1, A_2, \cdots, A_m \subseteq L$ such that $|A_i \cap A_j| = 1$.

By Fisher's inequality, m = number of points $\le |L|$, i.e., the number of lines is at least m.

Theorem 28.5 (Erdos-Ko-Rado)
Let $n \ge 2k$. If $A_1, \cdots, A_m \subseteq [n]$ such that

1. $|A_i| = k$
2. $|A_i \cap A_j| \ne 0$, $1 \le i < j \le m$

Then
$$m \le \binom{n-1}{k-1}$$

Proof by graph theory.