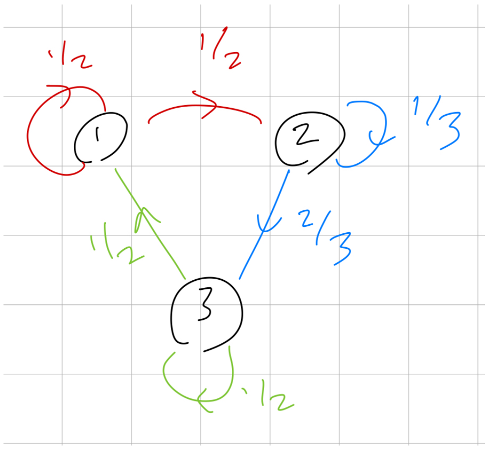
26 Exam 2 Review

1. (a) Let $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/3 & 0 \\ 0 & 2/3 & 1/2 \end{bmatrix}$ be the transition matrix of a Markov chain. Determine if it is regular.



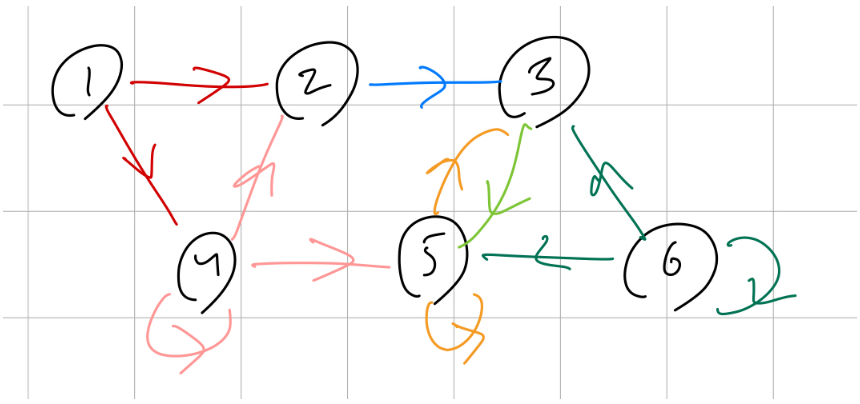
We can go from 1 to 2 to 3 to 1, meaning there is one communication class, so it is irreducible. Moreover, if we are in state 1, we can return in 1 or 2 steps, meaning $\text{GCD}(1, 2) = 1$.

Since the Markov chain is both irreducible and aperiodic, it is regular.

(b) Describe what P^{1000} looks like

We want to find the stable vector by solving $P\vec{q} = \vec{q}$, with $q_1 + q_2 + q_3 = 1$.
$\implies P^{1000} \approx \begin{bmatrix} 4/11 & 4/11 & 4/11 \\ 3/11 & 3/11 & 3/11 \\ 4/11 & 4/11 & 4/11 \end{bmatrix}$

2. (a) Find all communication classes and classify them as recurrent or transient:



$\{6\}$ is transient
$\{3, 5\}$ is recurrent
$\{1\}, \{4\}$, and $\{2\}$ are transient.

(b) There is more than one communication class, so it cannot be irreducible or regular.

(c) What is the period of state 3?

We can get back to it in 2 and 3 steps, so its period is 1.

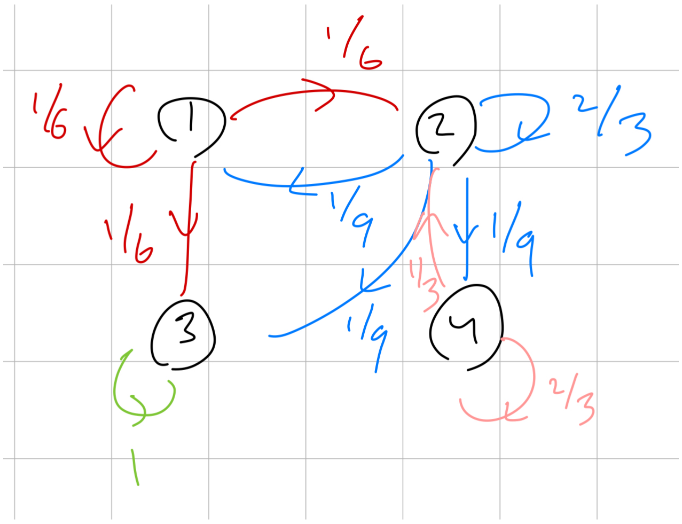
3. (a) A weighted, 10 sided die is rolled, where 5 has $\frac{1}{3}$ chance of showing. All other values have equal chance.
Using the fundamental matrix, determine the expected rolls until we get a 5.

Consider 2 states, representing whether we get a 5 or not. The following transition matrix has columns and rows that correspond to rolling a 5, and not, respectively.

$$P = \begin{bmatrix} 1 & 1/3 \\ 0 & 2/3 \end{bmatrix}$$

Here, $Q = \begin{bmatrix} 2/3 \end{bmatrix}$, which means $(I - Q)^{-1} = \begin{bmatrix} 3 \end{bmatrix}$. So, we expect to perform 3 rolls.

4. (a) Suppose we have the following Markov chain:



What is the probability that you reach state 2 before 3 if you begin in state 1?

We want to convert state 2 into an absorbing state. By switching states 1 and 3 to turn our transition matrix in canonical form, we get the following matrix (with rows and columns corresponding to states 3, 2, 1, 4):

$$\begin{bmatrix} 1 & 0 & 1/6 & 0 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix}$$

$$S = \begin{bmatrix} 1/6 & 0 \\ 2/3 & 1/3 \end{bmatrix} \qquad Q = \begin{bmatrix} 1/6 & 0 \\ 0 & 2/3 \end{bmatrix}$$

With columns and rows corresponding to 1 and 4,

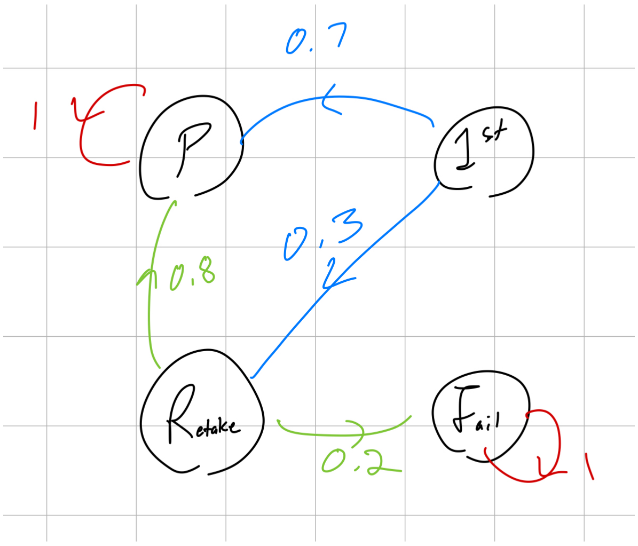
$$\implies (I - Q)^{-1} = \frac{18}{5} \begin{bmatrix} 1/3 & 0 \\ 0 & 5/6 \end{bmatrix}$$

$$\implies S(I - Q)^{-1} = \begin{bmatrix} 1/5 & 0 \\ 4/5 & 1 \end{bmatrix}$$

With rows corresponding to 3 and 2, and columns corresponding to 1 and 4.

Thus, there is a $\frac{4}{5}$ chance to reach state 2 before 3.

5. (a) In a math course, a student is either taking the class for the first time, has passed the class, is retaking the class, or failed twice and cannot retake.
If they are taking the class for the first time, they have a 70% chance of passing.
If they are taking the class for the second time, they have a 80% chance of passing.
If a student is taking the class for the first time, what is the probability he never passes?



With rows and colums corresponding to passing, failing, first, and retake:

$$\implies P = \begin{bmatrix} 1 & 0 & 0.7 & 0.8 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}$$

$$\implies S(I-Q)^{-1} = \begin{bmatrix} 0.7 & 0.8 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.3 & 1 \end{bmatrix} = \begin{bmatrix} 0.94 & 0.8 \\ 0.06 & 0.2 \end{bmatrix}$$

With rows corresponding to passing and failing, and columns corresponding to first and retake.

Thus, if it is the first time taking the class, there is a 6% chance that we don't pass.