25 Oddtown and Eventown

From last time: Let $A_1, \cdots, A_m \subseteq [n] = \{1, 2, \cdots n\}$ such that

1. $|A_i|$ is odd for $1 \leq i \leq m$
2. $|A_i \cap A_j|$ is even for $1 \leq i \neq j \leq m$

How large can m be?

Example 25.1
Take $n = 10$.

$\{\{2, 4, 6\}, \{10\}, \{1, 2, 4\}, \{5, 7, 9\}, 1, 4, 5, 6, 7\}\}$ is a possible candidate.

$\{\{1\}, \{2\}, \cdots, \{10\}\}$ is a construction that we can generalize to any n, which shows that $m \geq n$.

Definition 25.2
The **characteristic vector** (**incidence**) of a set A_i is defined by

$$\vec{v}_i(k) = \begin{cases} 1 & k \in A_i \\ 0 & k \notin A_i \end{cases}$$

Example 25.3
From earlier, if $A_3 = \{1, 2, 4\}$,

$$\vec{v}_3 = (1, 1, 0, 1, 0, 0, 0, 0, 0, 0)$$

And

$$A_5 = \{1, 4, 5, 6, 7\} \implies \vec{v}_5 = (1, 0, 0, 1, 1, 1, 1, 0, 0, 0)$$

Now what is the dot product?

$$\vec{v}_3 \cdot \vec{v}_5 = 2 = |A_3 \cap A_5|$$

In general, $\vec{v}_i \cdot \vec{v}_j = |A_i \cap A_j|$.

We treat the characteristic vector as lying in $\mathbb{F}_2^n$

Theorem 25.4 (Oddtown)
Let $A_1, \cdots, A_m \subseteq [n]$, and assume

1. $|A_i|$ is odd
2. $|A_i \cap A_j|$ is even

Then $m \leq n$.

Thus, the construction $\{\{1\}, \{2\}, \cdots, \{n\}\}$ shows that this is the best possible arrangement.

Proof. Let $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{F}_2^n$ be the characteristic vectors of these sets.

It suffices to show that $\vec{v}_1, \cdots, \vec{v}_m$ must be linearly independent over $\mathbb{F}_2^n$, which has dimension n.

Since the total number of linearly independent vectors cannot exceed the dimension of the space, the number of vectors m must be less than or equal to the dimension of $\mathbb{F}_2^n$, which is n.

Observe over $\mathbb{F}_2$, we know that our conditions imply that $|A_i| = \vec{v}_i \cdot \vec{v}_i = 1$, and $|A_i \cap A_j| = \vec{v}_i \cdot \vec{v}_j = 0$.

Let $a_1, a_2, \cdots, a_m \in \mathbb{F}_2$ such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$$

Taking the dot product with $\vec{v}_i$,

$$\vec{v}_i \cdot (a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_i\vec{v}_i + \cdots + a_m\vec{v}_m) = \vec{v}_i \cdot \vec{0} = \vec{0}$$

$$\implies a_1(\vec{v}_i \cdot \vec{v}_1) + a_2(\vec{v}_i \cdot \vec{v}_2) + \cdots + a_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + a_m(\vec{v}_i \cdot \vec{v}_m) = 0$$

The only dot product that does not turn into 0 is $\vec{v}_i \cdot \vec{v}_i = 1$.

$$\implies 0 + 0 + \cdots + a_i(1) + 0 + \cdots + 0 = 0 \implies a_i = 0$$

This holds for all $1 \leq i \leq m$, so all of the a_i's must be 0.

Thus, $\vec{v}_1, \cdots, \vec{v}_m$ is necessarily linearly independent over $\mathbb{F}_2^n$. The number of linearly independent vectors can not exceed the dimension of the space, so $m \leq n$. □

What if $|A_i|$ is even instead? (with $|A_i \cap A_j|$ still even)

Example 25.5
$n = 6$

$\{\varnothing, \{1,2\}, \{3,4\}, \{5,6\}, \{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$
Here, we have chosen 8 sets, so we can immediately see that the bound will not be the same.

Theorem 25.6 (Eventown)
Let $A_1, \cdots, A_m \subseteq [n]$ such that

1. $|A_i|$ is even, $1 \leq i \leq m$
2. $|A_i \cap A_j|$ is even, $1 \leq i \neq j \leq m$

Then, $m \leq 2^{\lfloor n/2 \rfloor}$.

Proof. Let $\vec{v}_1, \cdots, \vec{v}_m$ be the usual characteristic vectors.
Now, $|A_i \cap A_j| = \vec{v}_i \cdot \vec{v}_j = 0$, and $|A_i| = \vec{v}_i \cdot \vec{v}_i = 0$.

Let $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$. Reall that $S^\perp$ is the set of vectors orthogonal to everything in S.

If $\vec{v}_i, \vec{v}_j \in S$, then $\vec{v}_i \cdot \vec{v}_j = 0$ (any two vectors in S are orthogonal).
Thus, $\vec{v}_i, \vec{v}_j \in S^\perp \implies S \subseteq S^\perp$.

Now, let $W = \text{span}(S)$.
Then, $W \subseteq W^\perp$ (since $S \subseteq S^\perp$).

We have that $m = |S| \leq |W|$, so $|W|$ yields an upper bound on m.

How large can $|W|$ be?

$$\dim(W) + \dim(W^\perp) = n$$

□