24 Distance and Sets (Section 4.1)

24.1 Distance and Sets (Section 4.1)

Recall if V is an n-dimensional vector space, e.g. $\mathbb{R}^n$ or $\mathbb{F}_2^n$, and we have m linearly independent vectors, then $m \leq n$.

What is the maximum possible points in $\mathbb{R}^n$ such that the distance between any 2 (distinct) points is the same distance?

In $\mathbb{R}^2$, this would be three points, being the vertices of any equilateral triangle.
Similarly in $\mathbb{R}^3$, a tetrahedron would work.

This construction generalizes to a **regular simplex**.
This explicit example provides a <u>lower bound</u> on the maximum possible points. That is, we can have m points in $\mathbb{R}^n$, where we know $m \geq n+1$.

> **Theorem 24.1**
> The maximum points in $\mathbb{R}^n$ such that all pairwise distances are equal is at most $n+1$.

Proof. Let $\vec{v}_1 = (0, 0, \cdots 0), \vec{v}_2, \cdots, \vec{v}_{n+1}$ represent the points, and assume that

$$||\vec{v}_i|| = 1 \qquad n_1 \geq i \geq 2$$

and tat

$$||\vec{v}_i - \vec{v}_j|| = 1 \qquad 1 \leq i \neq j \leq n+1$$

Then,

$$\begin{aligned} 1 = ||\vec{v}_i - \vec{v}_j||^2 &= (\vec{v}_i - \vec{v}_j) \cdot (\vec{v}_i - \vec{v}_j) \\ &= \vec{v}_i \cdot \vec{v}_i - 2(\vec{v}_i \cdot \vec{v}_j) + \vec{v}_j \cdot \vec{v}_j \\ &= ||\vec{v}_i||^2 - 2(\vec{v}_i \cdot \vec{v}_j) + ||\vec{v}_j||^2 \\ 1 &= 2 - 2(\vec{v}_i \cdot \vec{v}_j) \end{aligned}$$

$$\implies \vec{v}_i \cdot \vec{v}_j = \frac{1}{2} \qquad i \neq j \geq 2$$

Assume now, to the contrary, that we can have $n+2$ such points.

Then, $\vec{v}_2, \vec{v}_3, \cdots, \vec{v}_{n+1}, \vec{v}_{n+2}$ must be linearly dependent (since we have $n+1$ vectors in $\mathbb{R}^n$).

So, there must exist a $c_2, c_3, \cdots, c_{n+2}$ not all zero such that

$$c_2\vec{v}_2 + \cdots + c_{n+2}\vec{v}_{n+2} = \vec{0}$$

Applying the dot product of $(c_2\vec{v}_2 + \cdots + c_{n+2}\vec{v}_{n+2})$ on both sides,

$$(c_2\vec{v}_2 + \cdots + c_{n+2}\vec{v}_{n+2}) \cdot (c_2\vec{v}_2 + \cdots + c_{n+2}\vec{v}_{n+2}) = \vec{0} \cdot (c_2\vec{v}_2 + \cdots + c_{n+2}\vec{v}_{n+2})$$

Observe that $(c_i\vec{v}_i) \cdot (c_i\vec{v}_i) = c_i^2(\vec{v}_i \cdot \vec{v}_i) = c_i^2||\vec{v}_i||^2 = c_i^2$. This simplifies the above to

$$c_2^2 + c_3^2 + \cdots + c_{n+2}^2 + 2\sum_{i \neq j} c_i c_j (\vec{v}_i \cdot \vec{v}_j) = 0$$

We computed earlier that $\vec{v}_i \cdot \vec{v}_j = \frac{1}{2}$. Thus, we have that

$$c_2^2 + c_3^2 + \cdots + c_{n+2}^2 + \sum_{i \neq j} c_i \cdot c_j = 0$$

$$\implies \frac{1}{2}\left[(c_2 + c_3 + \cdots + c_{n+2})^2 + c_2^2 + c_3^2 + \cdots + c_{n+2}^2\right] = 0$$

Necessarily, $c_2 = c_3 = \cdots = c_{n+2} = 0$ because we have a sum of squares, which contradicts the assumption that $c_2, \cdots c_{n+2}$ were not all zero. Thus, $n = 2$ points is impossible!

Thus, there can be at most $n+1$ points in $\mathbb{R}^n$ which all have the same pairwise distance.

□

Note that by the simplex construction, this theorem shows that $n+1$ is the best possible number of points.

Definition 24.2
A **set** is a collection of distinct, unordered elements.

We define $[n] = \{1, 2, \cdots n\}$.

A **subset** A of set S satisfies the property that if $x \in A$, then $x \in S$.We write $A \subseteq S$.

The size or **cardinality** (number of elements) of S is denoted $|S|$.

The set with no elements is the **empty set**, denoted $\varnothing$ (not $\{\varnothing\}$).

24.1.1 Oddtown

There are n people forming clubs. every club must have an odd number of members.
For any two distinct clubs, they must share an even number of members.

How many clubs can be formed?

Equivalently, if $A_1, \cdots, A_m \subseteq [n] = \{1, \cdots n\}$ are distinct subsets such that

1. $|A_i|$ is odd for $1 \leq i \leq m$
2. $|A_i \cup A_j|$ is even for $1 \leq i \neq j \leq m$

How large can m be?