23 Lights Out, Extremal Combinatorics

From last time, we had that given a configuration $\vec{b}$, we want to solve the system of equations $b_1 = x_1 + x_2 + x_6$, $b_2 = x_1 + x_2 + x_3 + x_7, \cdots$, where $\vec{x}$ is the move vector.

We need $\vec{b} \in \text{col}(A) = (\ker(A))^\perp$, and we found that $\dim(\ker(A)) = 2$, so $\dim((\ker(a))^\perp) = 23$.

Let $\vec{n}_1$ and $\vec{n}_2$ be the basis vectors of $\ker(A)$ from last time.

Suppose we have a solution $\vec{x}_1$ for $A\vec{x} = \vec{b}$.
Then

$$A(\vec{x}_1 + \vec{n}_1) = A\vec{x}_1 + A\vec{n}_1 = \vec{b} + \vec{0} = \vec{b}$$

Theorem 23.1
If the Lights Out configuration is solvable, then

$$\vec{x}_1, \vec{x}_1 + \vec{n}_1, \vec{x}_1 + \vec{n}_2, \vec{x}_1 + \vec{n}_1 + \vec{n}_2$$

are the 4 posible solutions.

Note 23.2
Remarks:

1. If the lights are all on, this is <u>always</u> solvable for any $m \times n$ grid.
2. The $n \times n$ boards that can always be solved for any configuration is $n = 1, 2, 3, 6, 7, 8, 10, \cdots$, you can visit this site to see the full seuqence.
3. The total possible solutions for $n = 4 \implies 16$, $n = 5 \implies 4$, $n = 9 \implies 256, \cdots$. The full list can be found here.
4. This puzzle has been extended to looking at "torus" versions, where the top buttons also toggle the lights on the bottom.

23.1 Extremal Combinatorics

Suppose you are given subsets $A_1, A_2, \cdots, A_m$ of $\{1, 2, \cdots, n\}$ such that

1. Every set contains an odd number of elements (size)
2. If we intersect any two distinct subsets, they must share an <u>even</u> number of elements.

What is the maximum possible number of subsets?

If we place points in $\mathbb{R}^n$, how many can we place so that the distance between any 2 points is always the same distance? (We want the maximum, which is why these are called "extreme")

In $\mathbb{R}^2$, the best we can do is 3 points, being the vertices of an equilateral triangle.