22 Parity Domination Problem

From last time, we are given a random pattern of on and off lights, can we push a sequence of buttons and turn all the lights off?

Observations:

1. No button needs to be pressed more than once. Pressing a button twice is equivalent to doing nothing at all.
2. The problem is equivalent to beginning with all the lights off, and pushing buttons to yield the initial configuration.

Definition 22.1
The **field of order 2**, denoted $\mathbb{F}_2$, is the set $\{0, 1\}$ with the standard addition and multiplication, such that $1 + 1 = 0$.

We define $\mathbb{F}_2^n$ as the set of vectors with n entries where each entry in $\mathbb{F}_2$ (0 or 1).

It can be shown that $\mathbb{F}_2^n$ satisfies the usual properties of an n-dimensional vector space.

We enumerate the squares of the 5×5 board starting from 1, and going from left to right, top to bottom.

We wish to instead solve the problem of starting with all buttons being off, and reaching the initial configuration (which we said was equivalent to the original problem).

Let $\vec{b} = (b_1, b_2, \cdots, b_{25}) \in \mathbb{F}_2^{25}$ be the **configuration vector**: entry i is 1 if the light is on, and 0 otherwise.

Let $\vec{x} = (x_1, x_2, \cdots, x_{25}) \in \mathbb{F}_2^{25}$ be the **solution vector**, where the ith entry is 1 if we push the button, 0 otherwise.

Thus, we are solving (over $\mathbb{F}_2$):

$$\begin{aligned} b_1 &= x_1 + x_2 + x_6 \\ b_2 &= x_1 + x_2 + x_3 + x_7 \\ &\vdots \\ b_{25} &= x_{20} + x_{24} + x_{25} \end{aligned}$$

Thus, we solve $A\vec{x} = \vec{b}$.
Recall, if A is $m \times n$,

$$\ker(A) = \{\vec{z} \in \mathbb{R}^n : A\vec{z} = \vec{0}\}$$

$$(\ker(A))^\perp = \{\vec{y} \in \mathbb{R}^n : \vec{z} \cdot \vec{y} = 0 \quad \text{for all } \vec{z} \in \ker(A)\}$$

Where $(\ker(A))^\perp$ is the orthogonal complement of $\ker(A)$.

Lemma 22.2
If A is symmetric ($A = A^T$), then $\mathrm{col}(A) =$ span of the columns of $A = \mathrm{row}(A) = (\ker(A))^\perp$

Here, a vector $\vec{z}$ is in the kernel of A if each row vector of A dotted with $\vec{z}$ is 0. But then, that means the row vectors of A, along with their linear combinations, dotted with $\vec{z}$, a member of the kernel, leads to 0, meaning every linear combination of the row vectors of A is in $(\ker(A))^\perp$.

Thus, given a configuration $\vec{b}$, we are solving $A\vec{x} = \vec{b}$, which is equivalent to checking of $\vec{b} \in \mathrm{col}(A) = \mathrm{row}(A) = (\ker(A))^\perp$ (the system $A\vec{x} = \vec{b}$ in Lights Out satisfies that A is symmetric).

Note that this will not hold in a non square grids since $A \neq A^T$.

Now solving, doing all computations $\mod 2$, we find the system $A\vec{x} = \vec{0}$ has 2 free variables, which implies that $\dim(\ker(A)) = 2$. One can then obtain an <u>orthogonal</u> basis for $\ker(A)$, $\{\vec{n}_1, \vec{n}_2\}$, where

$$\vec{n}_1 = (0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0)$$

$$\vec{n}_2 = (1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1)$$

Since we need $\vec{b} \in (\ker(A))^{\perp}$, we have the following:

Theorem 22.3
Given a configuration vector $\vec{b}$ in Lights Out, it is solvable if and only if $\vec{b}$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ (mod 2).

Note that $\dim(\ker(A)) = 2$, so being in $\mathbb{F}_2^{25}$, which has dimension 25, means that $\dim(\ker(A))^{\perp}) = 23$.

Anything that lies in $(\ker(A))^{\perp}$ is solvable, and of the form $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{23}\vec{v}_{23}$ (23 because there are 23 basis vectors) where $a_i \in \{0, 1\}$.

So, $(\ker(A))^{\perp}$ has $(2)(2)\cdots(2) = 2^{23}$ possible elements.
We are in $\mathbb{F}_2^{25}$, which has 2^{25} elements.

This means that $\frac{2^{23}}{2^{25}} = \frac{1}{4}$ of the possible configurations are solvable!