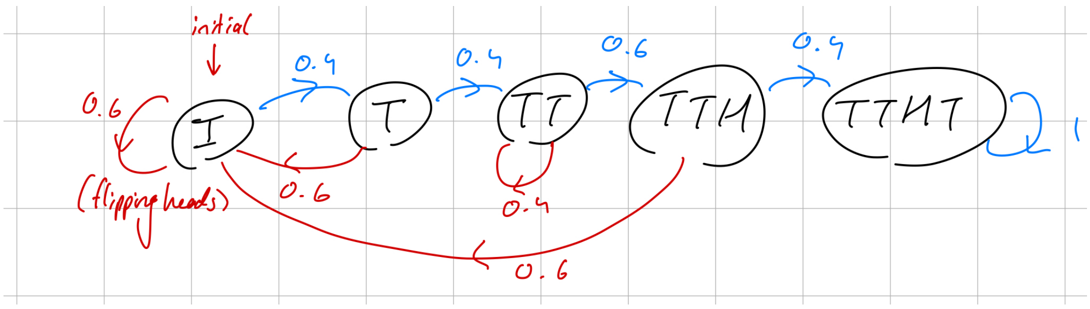
21 Markov Chain Examples

Example 21.1
A biased coin with probability 0.6 chance of heads is flipped.

What is the expected number of flips to get the pattern Tails, Tails, Heads, Tails?



The canonical form, with columns and rows corresponding to $TTHT$, TTH, TT, T, and I respectively:

$$\implies P = \begin{bmatrix} 1 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.4 \\ 0 & 0.6 & 0 & 0.6 & 0.6 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0.6 & 0 & 0.6 & 0.6 \end{bmatrix}$$

$$(I-Q)^{-1} = \begin{bmatrix} . & . & . & 2.5 \\ & & & 4.1667 \\ & & & 6.25 \\ & & & 15.265 \end{bmatrix}$$

Here, we want the column sum of the fundamental matrix that corresponds to starting in state I, which is ≈ 28.54 flips. So, we expect to perform 28.54 flips before achieving our desired pattern.

Example 21.2
A knight moves in an "L"-shape. A knight moves randomly on an 8×8 chessboard.

If it begins in the corner, how long will it take to return back to the corner?

Here, we enumerate the squares on the chessboard by starting in the top left and going left to right, top to bottom (top row is numbered $1, 2, \cdots 8$, next row is numbered $9, 10, \cdots, 16$, etc.).

There is only one communication class here. The "knight's tour" visits every square on the chessboard. This means that it is irreducible, so the stable vector exists.

Remember that the mean return time is the ith entry of the stable vector q_i.

The total possible number of knight moves on the board is 336.

Let d_i be the total possible moves in state i. $d_{17} = 4$, where square 17 is the entry $(3, 1)$ on the chessboard. Then $\frac{1}{d_i}$ is the chance to return to an appropriate square.

Note that the only way we can get to state 1 is to start in state 11 or state 18.

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{d_{11}} & \cdots & 0 & \frac{1}{d_{18}} & \cdots & 0 \\ \cdots \end{bmatrix} \begin{bmatrix} \frac{d_1}{\text{all possible on } 8 \times 8} \\ \frac{d_2}{\text{all possible on } 8 \times 8} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{2}{\text{all possible moves}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{d_1}{\text{all possible moves}} \\ \vdots \end{bmatrix}$$

We have found the stable vector! This satisfies $P\vec{q} = \vec{q}$.
We stated that $\frac{1}{q_i}$ is the expected number of transitions to return back to state i.
Thus, if we begin in the corner, we expect

$$\frac{1}{\frac{d_1}{\text{all moves}}} = \frac{1}{\frac{2}{336}} = 168 \text{ moves to return back}$$

21.1 The Game of Lights Out

This game originated in 1995 with a 5×5 grid of lights. Pushing a button switches the adjacent buttons on/off (not diagonal). Given a random pattern of on/off lights, can we push a sequence of buttons and turn them all off?

This game can be represented with a 5×5 grid of nodes in a graph, where all adjacent nodes have an edge between them. Colored in nodes are on, uncolored nodes are off.

In terms of graphs, we want to choose some vertices that "dominate" all other vertices, so that ON vertices are dominated by an ODD number of vertices, and OFF vertices an EVEN number of vertices.

This is called the "Parity Domination Problem"