20 Markov Chain Examples

From last time, we had that

$$P = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 1/3 \\ 1/2 & 1/3 & 1/3 & 0 \end{bmatrix}$$

If we begin in state 1, what is the probability we reach state 2 before state 4?
Here, we treat states 2 and 4 as absorbing.

$$P = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 1/3 & 1/2 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$$

Where rows and columns correspond to states 4, 2, 3, and 1. Here,

$$(I-Q)^{-1} = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \implies S(I-Q)^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Where rows correspond to states 4 and 2 (recurrent), and columns correspond to 3 and 1 (transient).

The bottom right entry of this matrix gives us the probabiliity that we reach state 2 before state 4, given that we start in state 1.

Example 20.1
In tennis, a game is at "deuce" if it is 40-40.

If player A is serving, winning the next point is an advantage ("ad in"). Otherwise player B has "ad out". The game ends when a player scores two points in a row. If A is serving, he has 60% chance of winning the point.

How long will the game take until it ends (from deuce)?

$$P = \begin{bmatrix} 1 & 0 & 0 & 0.6 & 0 \\ 0 & 1 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \end{bmatrix}$$

Where rows and columns correspond to A winning, B winning, deuce, ad in (player A since A is serving), and ad out (player B) respectively.

$$Q = \begin{bmatrix} 0 & 0.4 & 0.6 \\ 0.6 & 0 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}$$

$$(I-Q)^{-1} = \begin{bmatrix} 1.923 & 0.769 & 1.154 \\ 1.154 & 1.462 & 0.692 \\ 0.769 & 0.308 & 1.462 \end{bmatrix}$$

Where columns and rows correspond to deuce, ad in, ad out respectively.

At deuce, we expect $1.923 + 1.154 + 0.769 = 3.846$ plays until the game ends.

Example 20.2

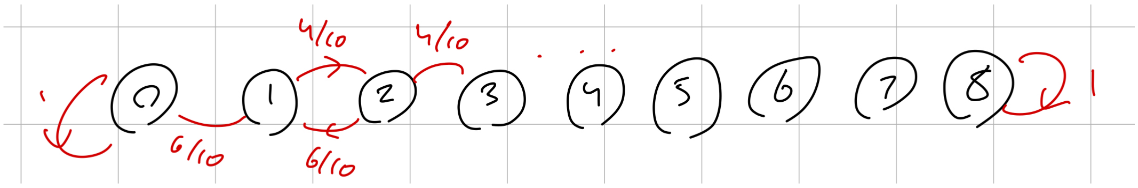
Bob has 3 dollars. He plays a game where if he bets x dollars, he wins x dollars with probability $\frac{4}{10}$, and loses with probability $\frac{6}{10}$. Bob quits when he loses everything or reaches 8 dollars (he doesn't try to get more than 8 dollars).

Bob decides to either

1. Bet 1 dollar each time
2. Go "all in" every time to get 8 dollars (but not over)

Which method should Bob use?

1. The corresponding Markov chain is a biased walk with absorbing boundaries:



In canonical form (with rows and columns corresponding to $0, 8, 1, \cdots, 7$):

$$P = \left[\begin{array}{cc|ccccccc} 1 & 0 & & & & & & & \\ 0 & 1 & & & & & & & \\ \hline 0 & 0 & & & & & & & \\ 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & & & & Q & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & & & & & & & \end{array}\right]$$

$$\implies (I - Q)^{-1} = \begin{bmatrix} . & . & 2.51 & . & . & . & . \\ & & 3.18 & & & & \\ & & 1.96 & & & & \\ & & 1.15 & & & & \\ & & 0.6 & & & & \\ & & 0.24 & & & & \\ & & 1.51 & & & & \end{bmatrix}$$

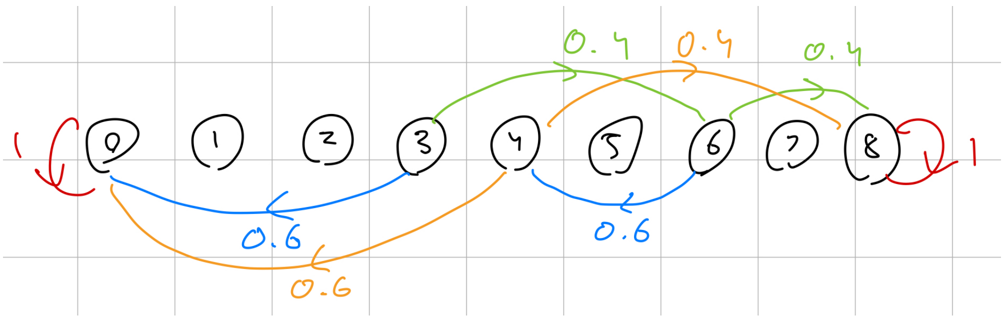
The column we wrote out corresponds to state 3.
The column sum is around 11.15, which is how long we expect the game to last. But we want the probability that we get to 8!

We find

$$S(I - Q)^{-1} = \begin{bmatrix} . & & & & & & \\ 0.02 & 0.051 & 0.096 & 0.165 & 0.268 & 0.422 & 0.653 \end{bmatrix}$$

So, there is around a 0.096 probability that we win 8 dollars when we start with 3.

2. The corresponding Markov chain for this strategy is as follows:



We see here that the only relevant states are $0, 8, 3, 4$, and 6.

$$\implies P = \left[\begin{array}{cc|ccc} 1 & 0 & 6/10 & 6/10 & 0 \\ 0 & 1 & 0 & 4/10 & 4/10 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6/10 \\ 0 & 0 & 4/1 & 0 & 0 \end{array}\right]$$

With corresponding states 0, 8, 3, 4, and 6.

$$\implies (I-Q)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 6/25 & 1 & 3/5 \\ 2/5 & 0 & 1 \end{bmatrix}$$

From the first column sum, we expect 1.64 plays.

$$\implies S(I-Q)^{-1} = \begin{bmatrix} 0.744 & 0.6 & 0.36 \\ 0.256 & 0.4 & 0.64 \end{bmatrix}$$

With rows corresonding to states 0 and 8, and columns corresponding to states 3, 4, and 6.
The bottom left entry of the matrix tells us that if we star twith 3 dollars, we have a 25.6% chance to get 8 dollars.

So, strategy 2 is better.