2 The Leontief Input-Output Method (Continued)

2.1 The Leontief Input-Output Method

From the previous lecture, we learned that the total production vector $\vec{x}$ is defined as $\vec{x} = A\vec{x} + \vec{d}$, where $\vec{d}$ is the open sector demand. We can find $\vec{x}$ with the equation $\vec{x} = (I - A)^{-1}\vec{d}$, only if $(I - A)^{-1}$ exists.

Example 2.1
Let

$$A = \begin{bmatrix} 0.1 & 0.5 & 0.3 \\ 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$

be a consumption matrix, and assume the total production of the industries is $\vec{x} = \begin{bmatrix} 10 \\ 15 \\ 5 \end{bmatrix}$. Which industry has the highest demand in the open sector?

Here, we simply want to find $\vec{d}$ using the equation $(I - A)\vec{x} = \vec{d}$.

$$A \begin{bmatrix} 10 \\ 15 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 11.5 \\ 1.5 \end{bmatrix} = \vec{d}$$

Thus, industry 2 has the highest demand at 11.5 units.

2.2 Invertibility in open and closed economies

Consider the case of $\vec{d} = \vec{0}$ (closed economy). If $(I - A)$ is invertible, then $(I - A)\vec{x} = \vec{0} \implies \vec{x} = \vec{0}$

If $(I - A)$ is not invertible, $(I - A)\vec{x} = \vec{0}$ implies that $\ker(I - A)$ is non trivial. (There must be some vector $\vec{x}$ such that $(I - A)\vec{x} = \vec{0}$).
In this case, we are able to find a non-trivial production vector $\vec{x}$ that leads to no surplus in any industry.

We can also write $(I - A)\vec{x} = \vec{0}$ in the form $\vec{x} - A\vec{x} = \vec{0}$, which implies that $A\vec{x} = \vec{x}$, which means that $\vec{x}$ is any eigenvector for eigenvalue $\lambda = 1$ ($A\vec{x} = \lambda\vec{x}$).
Thus any eigenvector yields a possible solution for the total production.

Thus, there will be infinite choices for $\vec{x}$ since you can scale the eigenvector by any nonzero constant ($A(c\vec{x}) = \lambda(c\vec{x})$, where $c \neq 0$)

What about an open economy ($\vec{d} \neq \vec{0}$)?
If $(I - A)^{-1}$ exists, then the solution is unique: $\vec{x} = (I - A)^{-1}\vec{d}$.

Example 2.2
Suppose $A = \begin{bmatrix} 0.7 & 0.6 \\ 0.4 & 0.2 \end{bmatrix}$

If we think of these values in terms of money, and look at the first column, we see that it takes $0.7 + 0.4 = 1.1$ dollars worth of inputs to produce 1.0 dollars of output for commodity 1. This is an unprofitable sector.

$I - A = \begin{bmatrix} 0.3 & -0.6 \\ -0.4 & 0.8 \end{bmatrix} \implies \det(I - A) = 0 \implies$ not invertible.

To get the total production $\vec{x}$ for an open economy, imagine the process in stages.
First, we create $\vec{d}$ commodities to satisfy demands of the open sector.

$$\vec{x} = \vec{d}$$

But we can't just produce those goods out of nowhere. To create the goods for $\vec{d}$, we must have used $A\vec{d}$ internally to create the goods. (Recall that $A\vec{x}$ is the internal consumption to create everything).

$$\vec{x} = \vec{d} + A\vec{d}$$

$\vec{d}$ is how much we had to create to make sure consumers are happy, but those resources also did not just magically appear: those $A\vec{d}$ resources also needed to be produced by the industry somehow, and as you might be able to tell, this process repeats on and on.
To create the $A\vec{d}$ goods, we needed to use $A(A\vec{d})$ internally, and to produce those $A(A\vec{d})$ goods, we needed to use $A(A(A\vec{d}))$ internally, and thus:

$$\begin{aligned}\vec{x} &= \vec{d} + A\vec{d} + A^2\vec{d} + \cdots \\ &= (I + A + A^2 + \cdots)\vec{d} \\ &= (I - A)^{-1}\vec{d}\end{aligned}$$

And so a natural question we wish to ask when this series converges. If $\vec{x}$ exists and is finite, we expect

$$\lim_{n\to\infty} A^n = 0 \text{ matrix}$$

To give us some intuition on this problem, we can use the following analogue from calculus

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} = (1-x)^{-1}, |x| < 1$$

And here, we can think of x as A, and 1 as I.

Theorem 2.3
Let A be $n \times n$ with non-negative entries. The following are equivalent:

1. $\lim_{n\to\infty} A^n = 0$ matrix
2. $(I - A)^{-1}$ exists
3. There exists a non-negative vector $\vec{x}$ such that $(I - A)\vec{x}$ is positive.

Moreover, one can show if A has all non-negative entries AND the column sums are less than 1 (no unprofitable sectors!), then $(I - A)^{-1}$ exists (we get a unique $\vec{x}$!)

We now consider the case of $(I - A)^{-1}$ existing (which is the more interesting question to look at).
How does a change in the external demand $\vec{d}$ affect the total production $\vec{x}$?

Suppose demand increases by 1 in industry 2: We know originally, $\vec{x}_{\text{OLD}} = (I - A)^{-1}\vec{d}$. Now,

$$\vec{x}_{\text{NEW}} = (I - A)^{-1}\left(\vec{d} + \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right) = (I - A)^{-1}\vec{d} + (I - A)^{-1}\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} = \vec{x}_{\text{OLD}} + \big(\text{column 2 of } (I - A)^{-1}\big)$$

So here, $(I - A)^{-1}$ is basically telling us how much of a change in production we need to meet a change in demand.

More generally, if the open sector demand of industry j increases by 1 unit, column j of $(I - A)^{-1}$ yields the amount the production must change for each industry.

For k units, simply scale column j by k.

Example 2.4
We had $A = \begin{bmatrix} 0.2 & 0.6 \\ 0.4 & 0.1 \end{bmatrix}$, $(I - A)^{-1} = \begin{bmatrix} 15/8 & 5/4 \\ 5/6 & 5/3 \end{bmatrix}$ (from last lecture).
The second column of $(I - A)^{-1}$ tells us that if $\vec{d}$ changes by requiring 1 additional unit of commodity 2, we need an additional $\frac{5}{4}$ of unit 1, and $\frac{5}{3}$ of unit 2 to produce it