19 Fundamental Matrix

Last time, we had that if all recurrent states are absorbing,

$$P = \left[\begin{array}{c|c} I & S \\ \hline 0 & Q \end{array}\right]$$

And that

$$P^k = \left[\begin{array}{c|c} I & S(I + Q + Q^2 + \cdots + Q^{k-1}) \\ \hline 0 & Q^k \end{array}\right]$$

And we know that

$$\lim_{k\to\infty} S(I + Q + Q^2 + \cdots + Q^{k-1}) = S(I - Q)^{-1}$$

Theorem 19.1
Let P be in canonical form, and assume all recurrent states are absorbing.

Let i be an absorbing state, j be a transient state.
Then the ijth entry of $S(I - Q)^{-1}$ is the probability of ending up in state i eventually (in the long run), given that we start in state j.

Example 19.2
From the maze example,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \end{bmatrix}$$

Where the columns and rows correspond to states 1, 5, 3, 4, and 2 respectively.

Here,

$$S = \begin{bmatrix} 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 \end{bmatrix}$$

We had

$$(I - Q)^{-1} = \begin{bmatrix} 3/2 & 3/4 & 3/4 \\ 1/2 & 5/4 & 1/4 \\ 1/2 & 1/4 & 5/4 \end{bmatrix}$$

$$\implies S(I - Q)^{-1} = \begin{bmatrix} 1/2 & 1/8 & 5/8 \\ 3/4 & 7/8 & 3/8 \end{bmatrix}$$

Where the columns correspond to states 3, 4, and 2, and the rows correspond to states 1 and 5 respectively.

Note that every column should sum to 100%. There will always be a 100 percent chance that we will get stuck somewhere in the long run.

From this matrix, we get that there is a $\frac{7}{8}$ chance that we end up in room 5, given that we start in room 4.

What about irreducible Markov chains?
Recall, if we started in state i, we expected $\frac{1}{q_i}$ transitions until returning back to state i, where q_i is the ith entry of the stable vector.

Example 19.3

$$P = \begin{bmatrix} 0 & 1/5 & 1/3 & 0 \\ 1/2 & 2/5 & 1/6 & 1/4 \\ 1/3 & 1/5 & 1/3 & 1/2 \\ 1/6 & 1/5 & 1/6 & 1/4 \end{bmatrix}$$

One can find the stable vector

$$\vec{q} = \begin{bmatrix} 15/88 \\ 5/16 \\ 57/176 \\ 17/88 \end{bmatrix}$$

From the first entry, we expect $\frac{88}{15}$ transitions to return back to state 1.

How could we get this value with what we've done in the section?

Here, we create a new absorbing state 5, which "represents" state 1. Have all transitions to state 1 go to state 5 instead, and once we reach state 5, we have "returned" back to state 1.

The new canonical form is

$$P = \begin{bmatrix} 1 & 1/5 & 1/3 & 0 & 0 \\ 0 & 2/5 & 1/6 & 1/4 & 1/2 \\ 0 & 1/5 & 1/3 & 1/2 & 1/3 \\ 0 & 1/5 & 1/6 & 1/4 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With columns and rows corresponding to states 5, 2, 3, 4, and 1 respectively.
Submatrix Q is

$$Q = \begin{bmatrix} 2/5 & 1/6 & 1/4 & 1/2 \\ 1/5 & 1/3 & 1/2 & 1/3 \\ 1/5 & 1/6 & 1/4 & 1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

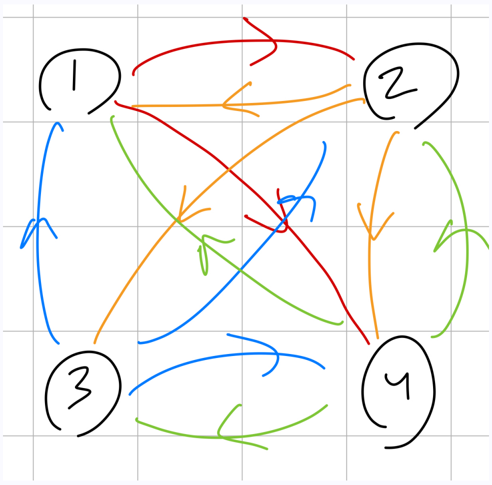
We find

$$(I - Q)^{-1} = \begin{bmatrix} & & & 11/6 \\ \vdots & \vdots & \vdots & 19/10 \\ & & & 17/15 \\ & & & 1 \end{bmatrix}$$

Where columns and rows correspond to state 2, 3, 4, and 1 respectively.

Recall that the ijth entry of $(I - Q)^{-1}$ represents the number of expected visits to transient state i given that we start in state j. So, the sum of column j is the number of expected visits to transient states before ending up in a recurrent state.
The column sum here is $\frac{88}{15}$, which is the expected number of transitions until we reach state 5 (i.e. returning back to state 1).

Example 19.4



1. If we begin in state 3, what is the expected number of times we visit state 1 before reaching state 2?

   Here, we want to treat state 2 to be an absorbing state.

$$P = \begin{bmatrix} & 0 & & \\ \vdots & 1 & \vdots & \vdots \\ & 0 & & \\ & 0 & & \end{bmatrix} \implies \begin{bmatrix} 1 & 1/2 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix}$$

   Where we swapped rows 2 and 1, meaning our canonical form has rows and columns corresponding to states 2, 1, 3, 4.

$$Q = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 0 & 0 & 1/3 \\ 1/2 & 1/3 & 0 \end{bmatrix} \implies (I - Q)^{-1} = \frac{1}{12}\begin{bmatrix} 16 & 8 & 8 \\ 3 & 15 & 6 \\ 9 & 9 & 18 \end{bmatrix}$$

   With columns and rows corresponding to states 1, 3, 4.

   So, we expect $\frac{8}{12}$ visits to state 1, if we start in state 3, before reaching state 2.

2. What is the expected number of transitions to reach state 2, given that we start in state 3?

   We take the column sum of $(I - Q)^{-1}$ corresponding to state 3, which gives us $\frac{8}{12} + \frac{15}{12} + \frac{9}{12} = \frac{8}{3}$ transitions.

3. If we begin in state 1, what is the probability we reach state 2 before 4?

   We now want to treat state 2 <u>and</u> 4 as absorbing. If we know the probability of getting stuck in state 2, then this means you never reached state 4, which is what we want.