18 Canonical Form, Fundamental Matrix

Recall from last time, we had the canonical form of a transition matrix P:

$$P = \left[\begin{array}{c|c} R & S \\ \hline 0 & Q \end{array}\right]$$

Where the left/upper sections correspond to recurrent states, and the right/lower sections correspond to transitive states.

Submatrix S will have at least one strictly positive entry (if there is no strictly positive entry, then all transitive classes go to other transitive classes, but then there must be a cycle so the classes would not be separate).

Thus, there must be a column in Q whose sum is strictly less than 1, meaning Q is not stochastic.

With this condition, we can show that $\lim_{k\to\infty} Q^k = 0$ matrix.

This means that in the long run, all transient states will eventually get stuck in a recurrent class.

Definition 18.1
The **indicator random variable**, denoted 1_e, of event e is

$$1_e = \begin{cases} 1 & \text{event } e \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

The expected value of an indicator random variable is $E(1_e) = 0\,\Pr(1_e = 0) + 1\,\Pr(1_e = 1) = P(\text{event occurs})$.

Let v_{ij} be the number of visits to transient state i, given that we start in transient state j, before reaching a recurrent class.

Define

$$1_e(k) = \begin{cases} 1 & \text{if we visit state } i \text{ on transition } k \text{ when starting from } j \\ 0 & \text{otherwise} \end{cases}$$

We want

$$\begin{aligned} E(v_{ij}) &= E\left(\sum_{k=0}^{\infty} 1_e(k)\right) \\ &= \sum_{k=0}^{\infty} E(1_e(k)) \\ &= \sum_{k=0}^{\infty} P(\text{visit transient state } i \text{ on the } k\text{th transition if start in } j) \\ &= \sum_{k=0}^{\infty} (Q^k)_{ij} \end{aligned}$$

This means that the ijth entry of $\sum_{k=0}^{\infty} Q^k$ is the expected number of visits to transient state i before reaching a recurrent class.

Recall for $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = (1-r)^{-1}$$

For matrices, if the eigenvalues satisfy $|\lambda| < 1$,

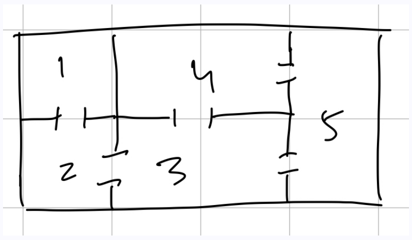
$$\sum_{k=0}^{\infty} Q^k = (I - Q)^{-1}$$

Which is called the **fundamental matrix** of the Markov chain.

Theorem 18.2
Let i, j be transient states. If Q is the submatrix in canonical form,

1. The ijth entry of $(I-Q)^{-1}$ is the expected number of visits to i before reaching a recurrent class, given we start in j.
2. The column sum of the jth column yields the expected number of transitions until reaching a recurrent class.

Example 18.3
Consider the maze



Where rooms 1 and 5 are traps.

If we begin in room 3, what is the expected number of transitions until we are trapped?

We have the transition matrix

$$P = \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1 \end{bmatrix}$$

We wish to put this matrix into canonical form. We do this by switching states 2 and 5.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \end{bmatrix}$$

Where the columns and rows correspond to states 1, 5, 3, 4, and 2 respectively.

The bottom right 3×3 submatrix corresponds to Q. So,

$$Q = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}$$

With columns and rows representing states 3, 4, 2 respectively.

$$(I-Q)^{-1} = \begin{bmatrix} 3/2 & 3/4 & 3/4 \\ 1/2 & 5/4 & 1/4 \\ 1/2 & 1/4 & 5/4 \end{bmatrix}$$

From $(I-Q)^{-1}$, we find that the column sum for the column corresponding to state 3 is $\frac{5}{2}$. Thus, if we start in room 3, we expect to perform 2.5 transitions until we are trapped.

Consider the case of recurrent classes only being absorbing states.
The canonical form would be

$$P = \left[\begin{array}{c|c} I & S \\ \hline 0 & Q \end{array}\right]$$

With this form, we can do the following:

$$P = \left[\begin{array}{c|c} I & S \\ \hline 0 & Q \end{array}\right]\left[\begin{array}{c|c} I & S \\ \hline 0 & Q \end{array}\right] = \left[\begin{array}{c|c} & S+SQ \\ \hline & Q^2 \end{array}\right] = P^2$$

$$\left[\begin{array}{c|c} I & S \\ \hline 0 & Q \end{array}\right]\left[\begin{array}{c|c} & S+SQ \\ \hline & Q^2 \end{array}\right] = \left[\begin{array}{c|c} & S+SQ+SQ^2 \\ \hline & Q^3 \end{array}\right] = P^3$$

So,

$$\lim_{k\to\infty} P^k = \left[\begin{array}{c|c} & S+SQ+SQ^2+\cdots \\ \hline & \end{array}\right] \implies S+SQ+SQ^2+\cdots = S(I+Q+Q^2+\cdots) = S(I-Q)^{-1}$$