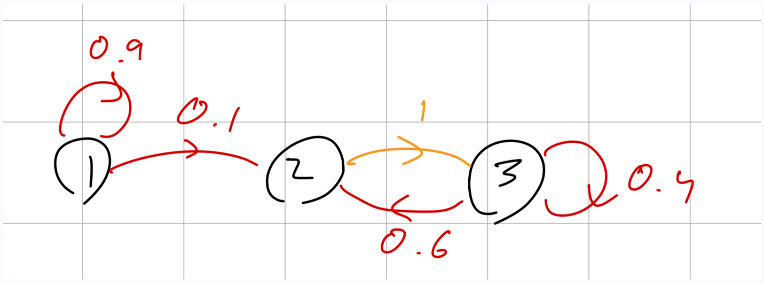
17 Periodicity, The Fundamental Matrix (Section 3.5)

17.1 Periodicity

Definition 17.1
The **period** of a state i of a Markov chain is the GCD (greatest common divisor) of all k such that entry i, i of P^k is positive.

Example 17.2
Suppose we have the following Markov chain:

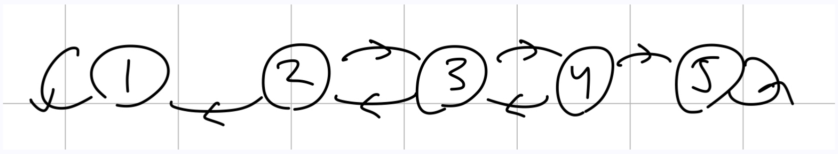


What is the period of state 2?

One path we can take to go from state 2 to state 2 is $2 \to 3 \to 2$, which is two steps.
We can also take the path $2 \to 3 \to 3 \to 2$, which is three steps.
There are also paths that take $4, 5, 6, \cdots$ steps.

Thus, the period is the $\text{GCD}(2, 3, 4, \cdots) = 1$.
Here, technically we already knew what the period is because when we found two paths with consecutive lengths (2 and 3), since we already know from that point that the GCD must be 1.

Example 17.3
Consider an unbiased walk on 5 states with absorbing boundaries.



What is the period of each state?

We can clearly see that in the boundary states, we can return back to them in any number of steps.
So, their periods are $\text{GCD}(1, 2, 3, \cdots) = 1$.

For states 2, 3, and 4 to return back, the number of times we go to the right must equal the number of times we go to the left.
Thus, the period is $\text{GCD}(2, 4, 6, 8, \cdots) = 2$.

Notice that here, the communication classes are $\{1\}, \{2, 3, 4\}, \{5\}$. We will see that every state in the same communication class must have the same period.

Theorem 17.4
Given a communication class C, the period for any state in C is the same.

Definition 17.5
If every state (or every class) has period 1, the Markov chain is **aperiodic**.

Theorem 17.6
A Markov chain is regular if and only if it is irreducible and aperiodic.

17.2 The Fundamental Matrix (Section 3.5)

Definition 17.7
A state i is said to be **absorbing** if entry (i, i) of the transition matrix ix 1.

Example 17.8
From the example in the last section,

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 1/3 & 1/3 \\ 1/2 & 1/2 & 1/3 & 1/3 \\ 0 & 1/4 & 1/3 & 1/3 \end{bmatrix}$$

Where the rows/columns are ordered as 1, 2, 3, 4.

We reorder the states so that recurrent states are first.
We had that $\{1\}$ is transient, and $\{2, 3, 4\}$ is recurrent.
So, we can rearrange the columns/rows as such (where rows/columns are ordered 2, 3, 4, 1):

$$P = \begin{bmatrix} 1/4 & 1/3 & 1/3 & 0 \\ 1/2 & 1/3 & 1/3 & 1/2 \\ 1/4 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Alternatively, instead of reordering the entire matrix, we can just switch states 1 and 4, because we just want transient states to be last. Here, the columns/rows are ordered as 4, 2, 3, 1.

$$P = \begin{bmatrix} 1/3 & 1/4 & 1/3 & 0 \\ 1/3 & 1/2 & 1/3 & 0 \\ 1/3 & 1/4 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

The ordering doesn't matter as long as transient states are last.

The upper left 3×3 submatrix is the chance of going from a recurrent state to a recurrent state.
The block of 3 elements below that submatrix is all 0, which represents the 0% chance of going from a recurrent to a transient state, which makes sense.

Abstracting the details away, we have a block matrix

$$\begin{bmatrix} P_1 & S \\ 0 & Q \end{bmatrix}$$

Where 0 is the 0-matrix. We know that this lower left block will always be a block of 0s, because we can never go from recurrent states to transient ones.

More generally, if P is the transition matrix with "sub transition matrices" $P_1, P_2, \cdots, P_k$ of recurrent classes $C_1, C_2, \cdots, C_k$, the **canonical form** of P is

$$P = \begin{bmatrix} P_1 & & & 0 \\ & P_2 & & \\ & & \ddots & \\ 0 & & & P_k \end{bmatrix}$$

$$P = \left[\begin{array}{cccc|c} P_1 & & & 0 & \\ & P_2 & & & S \\ & & \ddots & & \\ 0 & & & P_k & \\ \hline & & 0 & & Q \end{array}\right] = \left[\begin{array}{c|c} R & S \\ \hline 0 & Q \end{array}\right]$$

Here, the left side and upper side of the partitions represent recurrent states, while the right side and lower side of the partitions represent transient states.

We know that P is stochastic (columns sum to 1), so P^k is stochastic.

$$\begin{bmatrix} R & S \\ 0 & Q \end{bmatrix}$$

There must be a positive entry somewhere in S, meaning there is a transient state that eventually goes to a recurrent state.

So then, there must be a column in Q that has a sum less than 1. Then, Q is not stochastic. So, $Q^k \to 0$.