# 16 Communication Classes, Classification of States and Periodicity (Section 3.4)

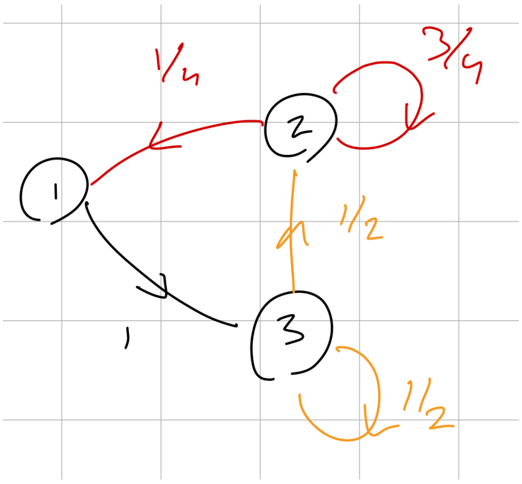
## 16.1 Communication Classes

From last time, we had that $X$ is a random variable on the number of transitions to go from $j$ to $i$ for the first time. We are interested in $E(X)$.

**Theorem 16.1**
For an irreducible Markov chain (1 communication class), if $X_{ij}$ is the random variable on the number of transitions to first visit state $i$ given that we start in state $j$, then $E(X_{ii}) = \frac{1}{q_i}$ is the mean return time, where $q_i$ is the $i$th entry of the stable/steady state vector.

**Example 16.2**
In the previous section, we had

$$P = \begin{bmatrix} 0 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 1 & 0 & 1/2 \end{bmatrix}$$




We found that $q = \begin{bmatrix} 2/14 \\ 8/14 \\ 2/7 \end{bmatrix}$.

By the theorem, we expect $\frac{1}{2/14} = 7$ to take transitions to return back to state 1, given that we start in state 1.

## 16.2 Classification of States and Periodicity (Section 3.4)

**Definition 16.3**
A communication class $C$ is a **transient class** if there is a state $j$ in $C$, and a state $i$ NOT in $C$ such that entry $(i, j)$ in $P^k$ is strictly positive for some positive integer $k$.

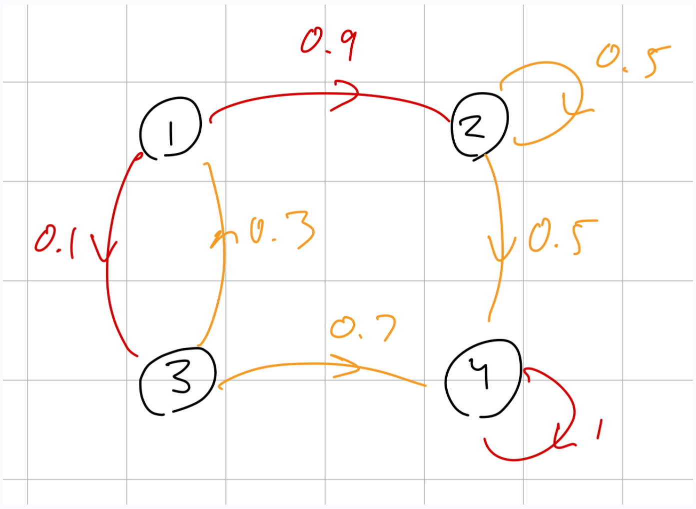
All states in $C$ are **transient states**.

Otherwise, they are **recurrent classes/states**.

**Note 16.4**

1. If the Markov chain is irreducible (one class), then all states must be recurrent.
2. Determine classes before classifying states.

**Example 16.5**
We are given the following Markov chain:




Classify the states as recurrent or transient.

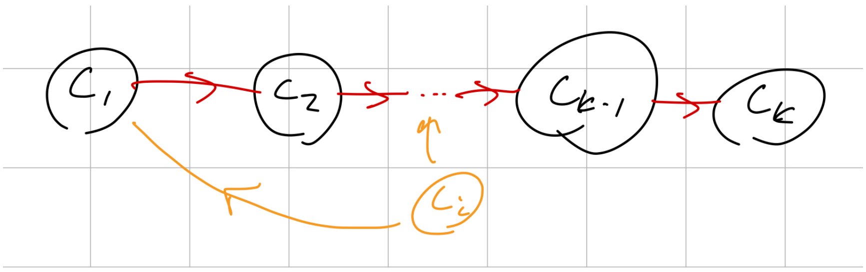
Here, the communication classes are $\{1, 3\}$, $\{2\}$, and $\{4\}$.

$\{4\}$ is recurrent since we are always stuck in 4.
There are clearly arrows coming out of the class $\{1, 3\}$, so this class must be transient.
There is also an arrow coming out of $\{2\}$, so it is also transient.

If we only have one class, then it is irreducible and must be recurrent. Now assume there are at least 2 transient classes $C_1$ and $C_2$.
Then, for some state in $C_1$, we can leave to some new class $C_2$.

Now in $C_2$, being transient, we must be able to transition to a different class, say $C_3$.
Otherwise, if states in $C_2$ transition out only to $C_1$, then this is a contradiction, since $C_1$ and $C_2$ should have ben 1 communication class.

We can generalize this:




For some class $C_i$, it must transition back to a previous communication class, $C_j$, contradicting $C_i, C_j$ being different classes. We have proven the following:

**Theorem 16.6**
A (finite) Markov chain must have at least one recurrent class.

**Example 16.7**

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 1/3 & 1/3 \\ 1/2 & 1/2 & 1/3 & 1/3 \\ 0 & 1/4 & 1/3 & 1/3 \end{bmatrix}$$

Here, we can see from the square submatrix starting from the 2nd rows and columns that $\{2, 3, 4\}$ is a class, because their entries are all positive.

We can also see from the first row of $P$ that we can not leave the class above, so this class is recurrent.

$\{1\}$ is in its own class, and because we can move to state 3, it is a transient class.

From here we could also say that $\{2, 3, 4\}$ must be recurrent by the theorem above.