13 Exam 1 Review

1. Let A be a consumption matrix for 3 industries where (with columns and rows corresponding to Agriculture, Fish, and Forest)

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.05 & 0.1 & 0.15 \\ 0.2 & 0 & 0.4 \end{bmatrix}$$

   (a) Which industry does agriculture depend on the most?

   We look at the highest value in the first column, which corresponds to 0.2 units of forest required to produce 1 unit of agriculture.

   (b) If the total output is 100 units of agriculture, 300 units of fish, and 200 units of forestry, how much was used internally?

$$\vec{d} = (I - A)\vec{x} \implies A\vec{x} = \vec{x} - \vec{d}$$

   We wish to find $A\vec{x}$:

$$A \begin{bmatrix} 100 \\ 300 \\ 200 \end{bmatrix} = \begin{bmatrix} 80 \\ 65 \\ 100 \end{bmatrix}$$

   (c) Now suppose the open sector demand is $\vec{d} = \begin{bmatrix} 648 \\ 216 \\ 432 \end{bmatrix}$. What is the total production?

   We would want

$$\vec{x} = (I - A)^{-1} \begin{bmatrix} 648 \\ 216 \\ 432 \end{bmatrix} = \begin{bmatrix} 1125 \\ 485 \\ 1095 \end{bmatrix}$$

   (d) Now suppose the open sector demand of agriculture decreased by 36 units. What is the new total production?

   Agriculture is industry 1, so we look at column 1 of $(I - A)^{-1} = \begin{bmatrix} 5/4 \\ 5/36 \\ 5/12 \end{bmatrix}$.

$$\vec{x}_{\text{new}} = \begin{bmatrix} 1125 + (-5/4) \cdot 36 \\ 485 + (-5/36) \cdot 36 \\ 1095 + (-5/12) \cdot 36 \end{bmatrix} = \begin{bmatrix} 1080 \\ 480 \\ 1080 \end{bmatrix}$$

2. Find an affinely dependent relation for the set of vectors $\{(1, 0, -2), (0, 1, 1), (-1, 5, 1), (0, 5, -3)\}$ (A linearly dependent relation AND sum of coefficients is 0).

   What if we added another vector $\vec{v}_5$ to the set? $\{(1, 0, -2), \cdots, \vec{v}_5\}$, with $\vec{v}_5 \in \mathbb{R}^3$
   We can quickly see that this set is also affinely dependent because with 5 vectors in $\mathbb{R}^3$, taking $\{\vec{v}_5 - \vec{v}_1, \cdots, \vec{v}_2 - \vec{v}_1\}$ we have 4 vectors in $\mathbb{R}^3$, meaning that the set must be linearly dependent, so the original set must also be affinely dependent.

   Generally, if we have $n + 2$ vectors in $\mathbb{R}^n$, we know they must form an affinely dependent set.

3. If $S = \{(x, y) : y = \sin x, x \in \mathbb{R}\}$, describe $\text{conv}(S)$ geometrically.

   $\text{conv}(S)$ is the region bounded between -1 and 1 on the y axis. It is a rectangle that expands outwards on the x axis infinitely.
   $\text{conv}(S)$ is the set $\{(x, y) : -1 \leq y \leq 1, -\infty < x < \infty\}$.

4. Suppose A is a subset of B, and B is convex. Is $\text{conv}(A)$ always a subset of B?

   First note that $B = \text{conv}(B)$ since B is convex.

If we have $\vec{p} \in \text{conv}(A)$, it is a convex combination of the points in A.
Because A is a subset of B, then $\vec{p}$ is a convex combination of points in B as well. Then, $\vec{p} \in \text{conv}(B)$, and $\text{conv}(B) = B$. So $\vec{p} \in B$, and so the statement is true.

5. Given A and B are convex sets, is the union $A \cup B$ always convex?

   No. If we have two disjoint convex sets A and B, their union will obviously not be convex.

   Let $A = \{(1,1)\}$, we can see A is convex. $B = \{(2,2)\}$, and B is also convex.
   But $A \cup B = \{(1,1),(2,2)\} \neq \text{conv}(A \cup B)$ which would be the line segment joining the two points.

6. Let $S = \{(-1,0),(2,3),(4,1)\}$. Does $(3,2)$ lie in $\text{aff}(S)$? If so, where does it lie relative to the triangle formed by the 3 points?

   $\vec{v}_2 - \vec{v}_1 = (3,3)$, $\vec{v}_3 - \vec{v}_1 = (5,1)$.
   $\{(3,3),(5,1)\}$ is linearly independent, so S is affinely independent. We can see that S is 3 vectors in $\mathbb{R}^2$, which means that $\text{aff}(S) = \mathbb{R}^2$, so $(3,2)$ must be in $\text{aff}(S)$.

   Alternatively, one can row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 2 & 4 & 3 \\ 0 & 3 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

   Thus the point $(3,2)$ lies <u>on</u> the triangle on the line segment joined by $(2,3)$ and $(4,1)$.

7. Remember for SVD that the trace is the sum of the eigenvalues, and the square of the singular values of Σ are the shared eigenvalues.