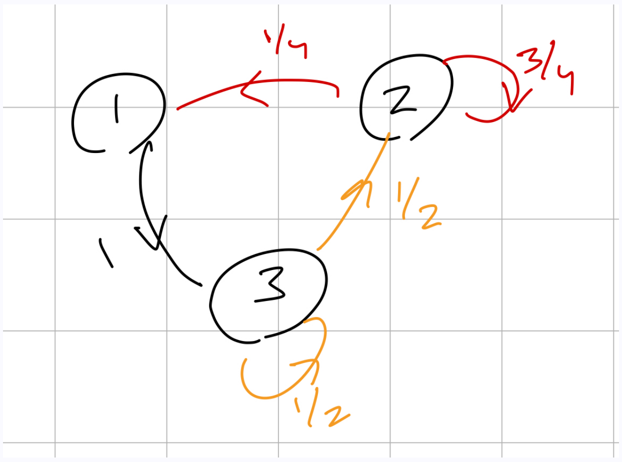
11 Markov Chains, Google PageRank

Note 11.1
One can prove if P is stochastic (the column vectors are all probability vectors - they sum to 1), then P^i will remain stochastic.

Example 11.2
Given the Markov chain



determine the probability of ending in state 1 in the long run.

Is the Markov chain regular? Here,

$$P = \begin{bmatrix} 0 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 1 & 0 & 1/2 \end{bmatrix}$$

One way we can determine regularity is by taking powers of P to see if all of the entries are ever all positive.

Alternatively, by the transition diagram, we can see that we are able to transition cyclically from states $1 \to 3 \to 2 \to 1$, and given a large enough number of transitions, we can begin anywhere, "stall" in state 2 via the self-loop, and then move to the final target state with positive probability, regardless of how unlikely it might be. Thus, it will always be possible to reach every state from any starting state.

So the transition matrix P is regular, so there is a unique stable vector $\vec{q}$.

To do so, we solve $(P - I)\vec{q} = \vec{0}$ and $q_1 + q_2 + q_3 = 1$.
Row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 1/4 & 0 & 0 \\ 0 & -1/4 & 1/2 & 0 \\ 1 & 0 & -1/2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/14 \\ 0 & 1 & 0 & 8/14 \\ 0 & 0 & 1 & 2/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And from the stable vector $\vec{q}$, we find that there is a $\frac{2}{14}$ chance to end in state 1 in the long run, regardless of the start state.

Example 11.3

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 3/4 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/4 \end{bmatrix} \implies P^{10} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0.087 & 0 & \cdots \\ 0 & 0.81 & 1 & \cdots \\ 0 & 0.054 & 0 & \cdots \\ 0 & 0.048 & 0 & \cdots \end{bmatrix}$$

Here, we can see that the original transition matrix states that both states 1 and 3 have only one transition to themselves with probability 1. That means if we start in either of those two states, we can never escape, which is an easy way to tell that P can not be regular.

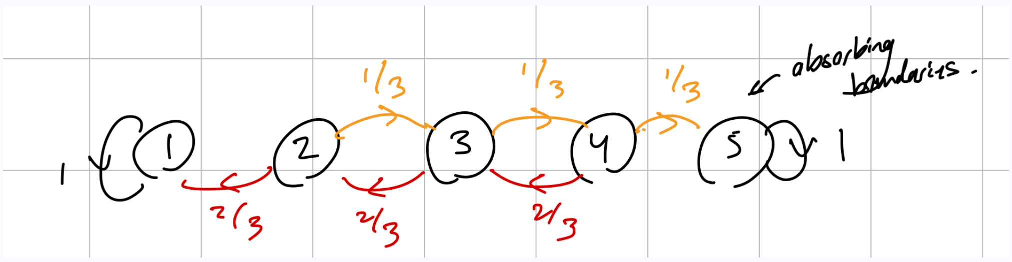
Also looking at P^{10}, we can see that P does not appear to be regular, because we would expect a regular matrix's columns to all converge to the same vector.

If we compute $P\vec{q} = \vec{q}$, $q_1 + q_2 + q_3 + q_4 + q_5 = 1$, and solve, we find:

$$\vec{q} = \begin{bmatrix} 1 - q_3 \\ 0 \\ q_3 \\ 0 \\ 0 \end{bmatrix} \qquad 0 \leq q_3 \leq 1$$

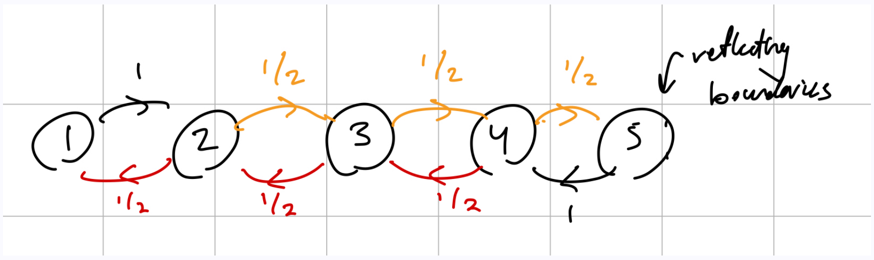
Thus the steady state vector $\vec{q}$ is not unique! Here, as mentioned earlier, if we begin in state 1, we can <u>never</u> reach state 2 in any number of transitions. Thus, it can not be regular.

Example 11.4
Consider a walk on 5 states, with probability $\frac{1}{3}$ moving to the right, $\frac{2}{3}$ to the left, with **absorbing boundaries**. The start and end points have 100% probability to return to itself.



Is this regular? No, it is not regular. If we start in state 1 or 5, it is impossible to go to any other state.

Now consider an **unbiased** walk on 5 states with **reflecting boundaries**: The endpoints have 100% chance to go back to the previous state.



Is this Markov chain regular? No, if we begin state 3, we can only return back to state 3 in an even number of steps, so it can not be regular.

However, if one solves $P\vec{q} = \vec{q}, q_1 + q_2 + q_3 + q_4 + q_5 = 1$, one gets a <u>unique</u> stable vector!

<u>**Thus, regularity is sufficient, but not necessary, to have a unique stable vector $\vec{q}$.**</u>

11.1 Google PageRank

We define a Markov chain as follows: The states are the webpages. Define a transition from page i to page j if there is a link from i to j. We then click webpages randomly with no bias.
If we obtain a unique stable vector (i.e. P^k stabilizes as $k \to \infty$), then the stable vector is used to rank the webpages based on popularity. i.e. the web page that we land on with the highest probability will rank the highest because there are more people linking to it.