10 Markov Chains, Stable/Steady-State Vectors (Section 3.2)

From last time, we had that $\vec{x}_2$ is the state vector after 2 transitions:

$$\vec{x}_2 = P^2\vec{x}_0 = \begin{bmatrix} 0.46 \\ 0.54 \end{bmatrix}$$

Where $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents us starting in the W state.
$\vec{x}_2$ is the first column of P^2, which is the probability of ending in W and L given that we start in W.
We use $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies P^2\vec{x}_0$ in the case that we start off in the loss state, which simply gives us the second column of P^2.

Theorem 10.1
If P is the transition matrix of a Markov chain, entry ij of P^k is the probability of ending in state i in k transitions, given that you begin state j.

Example 10.2

$$P = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 0 & 1/3 & 1/6 \\ 1/2 & 1/3 & 7/12 \end{bmatrix}$$

Where should we begin if we hope to end in state 3 after 3 transitions?

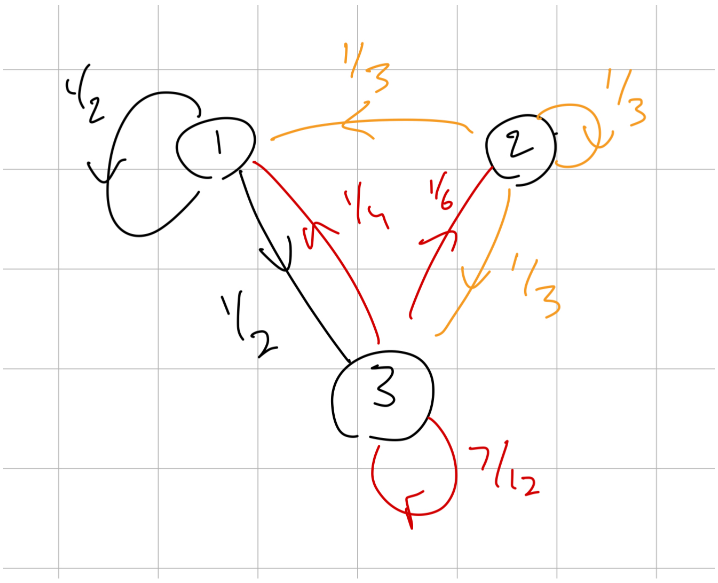
We find

$$P^3 = \begin{bmatrix} 0.35 & 0.35 & 0.34 \\ 0.12 & 0.14 & 0.14 \\ 0.53 & 0.51 & 0.52 \end{bmatrix}$$

We want the highest chance of ending in state 3 (which corresponds to the third row), and the first column contains the highest probability, which means we should start in state 1 to maximize our chance of ending in state 3.

We model Markov chains with a **transition diagram**.

In the last example, the transition diagram would look as follows:



10.1 Stable/Steady-State Vectors and PageRank (Section 3.2)

What happens to the Markov chain in the long run?

Definition 10.3
The **steady-state/stable vector** of the transition matrix P is the probability vector $\vec{q}$ such that:

$$P\vec{q} = \vec{q}$$

i.e. eigenvectors with eigenvalue 1. (Recall earlier that $P\vec{x}_0 = \vec{x}_1$)

Example 10.4
Game example:

$$\begin{matrix} & W & L \\ \begin{matrix} W \\ L \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{pmatrix}} \end{matrix} = P$$

If we solve $P\vec{q} = \vec{q}$, we have

$$\begin{aligned} P\vec{q} - \vec{q} &= \vec{0} \\ (P - I)\vec{q} &= \vec{0} \\ \left(\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -0.4 & 0.25 \\ 0.4 & -0.25 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We see here that the matrix on the left is linearly dependent since the columns/rows are multiples of each other, so might be infinite pairs of q_1, q_2 that fulfill this equation.

Thus, we need to add an extra constraint on q_1 and q_2: we need the probabilities to sum to 1, i.e. $q_1 + q_2 = 1$. Note that this is similar to the idea behind a convex combination.

So, we want to solve both $(P - I)\vec{q} = \vec{0}$ AND $q_1 + q_2 = 1$.

We row reduce

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.4 & 0.25 & 0 \\ 0.4 & -0.25 & 0 \end{array}\right]$$

Where the first row corresponds to $q_1 + q_2 = 1$. We get

$$\implies \left[\begin{array}{cc|c} 1 & 0 & 5/13 \\ 0 & 1 & 8/13 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{q} = \begin{bmatrix} 5/13 \\ 8/13 \end{bmatrix}$$

Note that

$$\begin{aligned} P\vec{q} &= \vec{q} \\ P(P\vec{q}) = P\vec{q} &= \vec{q} \\ P^2\vec{q} &= \vec{q} \end{aligned}$$

$$P^{1000} \approx \begin{bmatrix} 5/13 & 5/13 \\ 8/13 & 8/13 \end{bmatrix}$$

This is telling us that it doesn't matter which state we begin with, the chance of ending up in state 1 or 2 is always going to be the same.
We expect the columns to stabilize towards $\vec{q}$.

In particular, in the long run, the start state does NOT matter! Regardless of which state we start at, the chance to end in state 1 or 2 is always the same!

Definition 10.5

A stochastic matrix P is **regular** if P^k has all non-zero entries for some positive integer k.

Example 10.6

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

$$\implies P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \implies P^3 = P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

We can see here that P^k will never have all non-zero entries, so it is not regular.

In other words, a Markov chain is regular if after a certain number of transitions, there is a positive probability to move between any two states.

Theorem 10.7

Let P be the transition matrix of a regular Markov chain with n states. Then,

1. The coluns of P^k, $k \to \infty$ stabilize to a unique vector $\vec{q}$.
2. For any initial state vector $\vec{x}_0$,
$$\lim_{k\to\infty} P^k \vec{x}_0 = \vec{q}$$
i.e. the probability of ending in a specific state does NOT depend on the starting state.