1 The Leontief Input-Output Method

1.1 Topics of MATH401

1. Leontief Input/Output Method
2. Affine geometry and Graphics
3. Markov chains
4. Sets and graph theory

1.2 Section 1: The Leontief Input-Output Method

Suppose there are n sectors (mining, fishery, etc), each creating a commodity.
Each industry depends on the other industries' commodities.
Moreover, commodities may be used in external sources that don't produce goods in return.
This is called the **open sector**.

Definition 1.1
We let $\vec{d}$ denote the (open) **demand vector**, a column vector whose ith entry is the amount of commodity i that the open sector demands.

We say the economy is **open** if $\vec{d} \neq \vec{0}$ (else closed).

Definition 1.2
We relate the demands among the n industries by an $n \times n$ matrix A, where a_{ij} = amount needed of commodity i to create 1 unit of commodity j.

This is called the **consumption matrix**.

So, if $a_{37} = 0.1$, we need 0.1 of commodity 3 to make 1 unit of commodity 7.

Example 1.3
Take the following matrix with columns and rows corresponding to Steel, Coal, Electric, Iron

$$\begin{bmatrix} 0.1 & 0 & 0.3 & 0 \\ 0.4 & 0.2 & 0 & 0 \\ 0.05 & 0.1 & 0.1 & 0.3 \\ 0.01 & 0.1 & 0 & 0.1 \end{bmatrix}$$

Here, a_{13} denotes that we need 0.3 units of steel to make 1 unit of electric.

What is the largest consumer of coal? To answer this question, we would look at the coal row (2nd row) and see which column has the highest value. Here, steel uses the most coal out of all of the industries (0.4 units per unit of steel), and thus is the largest consumer.

What industry does iron depend on the most? We would have to look at the iron column (4th column), and we can see that we depend on electric the most, we need 0.3 units of it to make a unit of iron.

Definition 1.4
The **total output/production vector** $\vec{x}$ is a vector whose ith entry x_i = total output of commodity i needed to meet the demands among industries and the open sector (e.g. consumers)

Suppose we have that $a_{ij} = 0.2$, $x_j = 50$.

We know that a_{ij} = amount of i needed for 1 unit of j, so $a_{ij}x_j$ is the amount of commodity i needed to produce x_j units of commodity j.

Thus to meet <u>all</u> demands, we want total output of industry i to be x_i, where

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + d_i$$

Here, note that $a_{i1}x_1$ represents how much of i we need to satisfy 1, and d_i represents the open sector demand for i.

To meet the demands of all industries, we solve the equations

$$\begin{aligned} x_1 &= \sum_{j=1}^{n} a_{1j}x_j + d_1 \\ x_2 &= \sum_{j=1}^{n} a_{2j}x_j + d_2 \\ &\vdots \\ x_n &= \sum_{j=1}^{n} a_{nj}x_j + d_n \end{aligned}$$

which leads us to

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$\implies \vec{x} = A\vec{x} + \vec{d} \implies \vec{d} = (I - A)\vec{x}$$

We say $A\vec{x}$ is the **internal sector demand**. we would want the internal demand to be smaller compared to how much we are giving to consumers, we dont want to waste that many resources just creating things for the sake of creating other things.

Example 1.5

Given the consumption matrix

$$A = \begin{bmatrix} 0.2 & 0.6 \\ 0.4 & 0.1 \end{bmatrix}$$

and open sector demand vector

$$\vec{d} = \begin{bmatrix} 24 \\ 12 \end{bmatrix}$$

determine the total output needed of each commodity.

Here, we want to find $\vec{x}$ in $(I - A)\vec{x} = \vec{d}$. Thus, we find $(I - A)^{-1} = \begin{bmatrix} 15/8 & 5/4 \\ 5/6 & 5/3 \end{bmatrix}$, so

$$\vec{x} = (I - A)^{-1}\vec{d} = \begin{bmatrix} 15/8 & 5/4 \\ 5/6 & 5/3 \end{bmatrix} \begin{bmatrix} 24 \\ 12 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

Thus, we need 60 units of commodity 1, and 40 of commodity 2 to satisfy all demands.
Observe $\vec{x} - \vec{d} = \begin{bmatrix} 60 \\ 40 \end{bmatrix} - \begin{bmatrix} 24 \\ 12 \end{bmatrix} = \begin{bmatrix} 36 \\ 28 \end{bmatrix}$ is the amount of each commodity that was consumed internally by each industry. Here, this economy is not very efficient because over half of commodity 1 needed to be used internally