5 Breadth First Search, Bipartiteness

- BFS (connectivity, distances, bipartiteness)
- DFS
- Topological sorting
- Next: Greedy Algorithms

5.1 Breadth First Search

BFS partitions the graph into **layers** L_j such that

$$\begin{aligned} L_0 &= \{s\} \\ L_{j+1} &= \text{neighbors of vertices in } L_j \text{ that are not already neighbors of vertices in } L_0 \cup L_1 \cup \cdots \cup L_{j-1} \end{aligned}$$

Lemma 5.1
A vertex in L_j has distance j from s.

Proof. By induction on j.
Clearly, $L_0 = \{s\}$ is the set of vertices at distance 0.

If $L_0, L_1, \cdots, L_j$ satisfy the lemma, then vertices in L_{j+1} must have distance at least $j+1$ from s (otherwise, they would be in $L_0, L_1, \cdots,$ or L_j instead).
But there exists a path of length $j+1$ to all vertices in L_{j+1} since they are adjacent to vertices in L_j.

Vertices in L_{j+1} therefore satisfy the claim, so the claim follows for all j by induction. $\square$

5.1.1 Running Time

What is the running time of BFS? (in terms of $n = |V|$ and $m = |E|$)

We loop over all u in V; for each, do $O(\deg(u) + 1)$ operations.
Total running time: $O\left(\sum_{u \in V} (\deg(u) + i)\right) \subseteq O(m + n)$

Note that $m + n$ is the size of the adjacency list representation of a graph.

5.2 Bipartiteness

Lemma 5.2
Let T be a BFS tree of graph G, and let u and v be vertices of T with u in layer L_i, and v in layer L_j. Suppose $\{u, v\}$ is an edge of G. Then, $|i - j| \leq 1$.

Proof. Suppose without loss of generality that u is active before v.
We consider two cases:

1. v is in the tree when u becomes active. Then $\{u, v\}$ is a non-tree edge, and $pr[v]$ joined the tree before u, so $j = \text{layer}(pr[v]) + 1 \leq i + 1$
2. v is not in the tree when u becomes active. Then $\{u, v\}$ is a tree edge, and $j = i + 1$.

$\square$

Definition 5.3
A graph $G = (V, E)$ is **bipartite** if there is a partition (A, B) of V ($V = A \cup B, A \cap B = \varnothing$) such that for all $\{u, v\} \in E$, either $u \in A$ and $v \in B$ or $u \in B$ and $v \in A$.

Theorem 5.4
A connected graph with BFS tree T is bipartite if and only if there is no non-tree edge joining vertices in the same layer of T.

Proof. ($\Longleftarrow$) Consider the bipartition $A = L_0 \cup L_2 \cup \cdots$ and $B = L_1 \cup L_3 \cup \cdots$.
This shows the graph is bipartite since all non-tree edges join vertices whose layers differ by 1. (Also note that tree edges always join vertices in adjacent layers)

($\Longrightarrow$) Suppose that G has a non-tree edges $\{u, v\}$ with u and v in the same layer.
Proof continues next lecture.

□