28 Center Selection, Randomized Algorithms

- Approximation algorithms: Center selection
- Randomized algorithms

28.1 Center Selection

We found an algorithm last time that produces a covering radius of $2r$ if r is the optimal covering radius, but what if the optimal covering radius is unknown?

One approach uses binary search. Instead, by choosing the centers more carefully, we cna simplly remove the need to know the covering radius.

```
GreedyCover(S):
    if k >= |S|, return C = S
    select any s in S, and let C = {s}
    while |C| < k
        select s in S that maximizes dist(s, C) = min c in C dist(s, c)
        i.e. dist(s, C) is the distance between s and the closest center
         that has been chosen
    endwhile
    return C
```

Theorem 28.1
GreedyCover(S) returns a set of k points with covering radius at most twice the optimal radius.

Proof. Assume for a contradiction that we get k points with covering radius more than wr, where r = optimal radius.

Let s be a site at distance more than $2r$ from all centers.
Suppose the algorithm has centers C' and adds center c'.
Then c' is at least distance $2r$ away from all sites in C' since $\text{dist}(c', C') \geq \text{dist}(s, C') \geq \text{dist}(s, C) > 2r$.

So the algorithm implements the first k iterations of GreedyCover(S, r).
But that algorithm would have $S' \neq \varnothing$ after choosing k centers since $s \in S'$, so it would conclude that k centers cannot have radius r, a contradiction. □

28.2 Randomized Algorithms

28.2.1 Review of Probability Theory

Definition 28.2
A **finite probability space** consists of a **sample space** Ω (a finite set). For each **outcome** $i \in \Omega$, there is a **probability** $p(i) \geq 0$ with $\sum_{i\in\Omega} p(i) = 1$.

An **event** is a subset $\mathcal{E} \subseteq \Omega$, we define $\Pr(\mathcal{E}) = \sum_{i\in\mathcal{E}} p(i)$.

Example 28.3
For a fair coin, $\Omega = \{H, T\}$, $\Pr(H) = \Pr(T) = \frac{1}{2}$

Example 28.4
For n fair coins, $\Omega = \{H, T\}^n$, $\Pr(i) = \frac{1}{2n}$.
Then $\Pr(k \text{ heads}) = \frac{1}{2n} \cdot \binom{n}{k}$

Definition 28.5
For two events $\mathcal{E}$ and $\mathcal{F}$, the **conditional probability** of $\mathcal{E}$ given $\mathcal{F}$ is

$$\Pr(\mathcal{E}|\mathcal{F}) = \frac{\Pr(\mathcal{E} \cap \mathcal{F})}{\Pr(\mathcal{F})}$$

Definition 28.6
A (real-valued) **random variable** is a function $X : \Omega \to \mathbb{R}$.

Example 28.7
For n fair coins, the number of heads is a random variable.

Definition 28.8
The **expectation** of a random variable X is

$$E[x] = \sum_{i \in \Omega} p(i) X(i)$$

Example 28.9
The expected number of heads for n fair coins is

$$\sum_{i \in \{H,T\}^n} \frac{1}{2^n} \cdot \text{number of heads} = \sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} k = \frac{n}{2}$$

Fact: Expectation is Linear. $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$ even if X and Y are correlated.