25 Intractability (NP-completeness of 3-SAT, NP-complete problems)

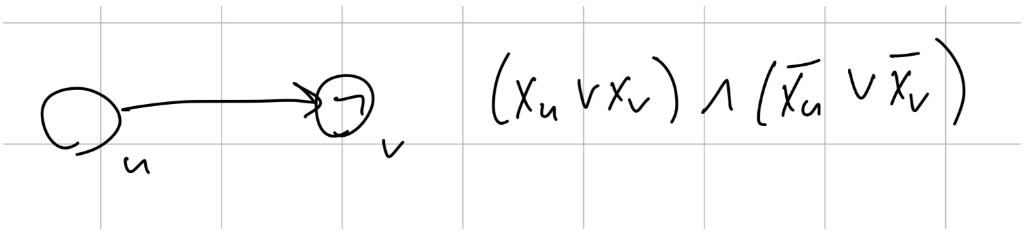
- Intractability
  - NP-completeness of 3-SAT
  - Hamiltonian cycle
- Rest of course: approximation algorithms and randomized algorithms

Proof. (That 3-SAT is NP-complete)

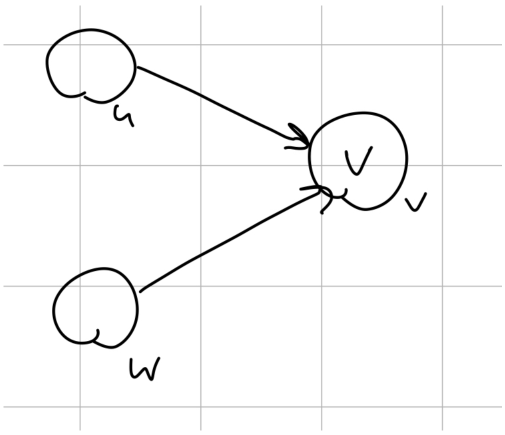
3-SAT is in NP since we can provide a satisfying assignment as the proof. This can be efficiently verified by checking that each clause evaluates to true.

Given any instance of Circuit SAT, we construct an instance of 3-SAT (so that one is satisfiable iff the other is).

Introduce a variable x_v for every vertex v of the circuit.
In the case of a negation gate, we add the following clauses:



The satisfiability of the circuit is in one to one correspondence with the satisfiability of the clauses.



We also claim that the or gate above corresponds to the clauses $(x_v \vee \overline{x}_u) \wedge (x_v \vee \overline{x}_w) \wedge (\overline{x}_v \vee x_u \vee x_w)$.

And for and gates: $(x_v \vee \overline{x}_u \vee \overline{x}_w) \wedge (\overline{x}_v, x_u) \wedge (\overline{x}_v \vee x_w)$.

A 0 input corresponds to $\overline{x}_v$, a 1 input corresponds to x_v, a free input corresponds to nothing, and an output corresponds to x_v.

To make this into a 3SAT formula, we add variables z_1, z_2, z_3, z_4 with clauses $(\overline{z}_i \vee z_3 \vee z_4) \wedge (\overline{z}_i \vee z_3 \vee \overline{z}_4) \wedge (\overline{z}_i \vee \overline{z}_3 \vee z_4) \wedge (\overline{z}_i \vee \overline{z}_3 \vee \overline{z}_4)$ for $i = 1$ and $i = 2$.
This forces z_1 and z_2 to be false in any 3SAT problem that contains these 8 clauses.

Then we can pad clauses with 1 or 2 terms to have exactly 3 terms by adding z_1 and z_2 to them. For example, we can pad x_v to be $x_v \vee z_1 \vee z_2$, and the entire clause will be true if and only if x_v is true. $\square$

25.1 Other NP-complete problems

- Travelling salesperson problem: given cities $v_1, \cdots, v_n$, distances $d(v_i, v_j)$, and a bound D, is there a tour of the cities with total length $\leq D$?
- Hamiltonian Cycle: is there a cycle that visits each vertex of a given graph exactly once?
- 3-dimensional matching: given disjoint sets X, Y, Z, each of size n, and a set $T \subseteq X \times Y \times Z$ of ordered triples, is there a subset of triples within T so that every element of $X \cup Y \cup Z$ is included in exactly one triple?
- 3-coloring problem: can we assign 3 colors to the vertices of a graph so that no two adjacent vertices have the same color?
- Subset sum: given numbers $w_1, \cdots, w_n$ and a target W, is there a subset of the w_is that sums to W?
- Scheduling with release times and deadlines: given n jobs indexed by $i \in \{1, \cdots, n\}$.
  job j has release time r_i, deadline d_i, duration t_i.
  job i must be scheduled for duration t_i, starting after r_i, and ending before d_i.

25.1.1 Example: Hamiltonian Cycle

Definition 25.1
Given a digraph $G = (V, E)$, a **hamiltonian cycle** is a cycle that contains each vertex.