22 Max Flow Extensions

- Max flow extensions
- Intractability: Reductions, NP, NP-completeness

22.1 Max Flow Extensions

$$\sum_v d_v = \sum_v (f^{in}(v) - f^{out}(v)) = 0$$

We will reduce this demand network into a standard max flow problem:
Constrct a graph G'

- Add source s^* and t^*
- For all $v \in S$, add edge (s^*, v) of capacity $-d_v$
- For all $v \in T$, add edge (v, t^*) of capacity $+d_v$
- We keep all edges between vertices of G with the same capacities.

Let $D = \sum_{v \in T} d_v = -\sum_{v \in S} d_v$

Theorem 22.1
There is a feasible circulation with demands $\{d_v\}$ iff the max $s^* - t^*$ flow in G' has value D. If all capacities and demands are integers, and the circulation is feasible, then it can be chosen to be integer-valued.

Proof. The max flow cannot exceed D since

$$\sum_{e \text{ out of } s^*} c_e = D$$

If there exists a feasible circulation in G, we can construct a flow in G' by sending flow $-d_v$ on edges (s^*, v) and flow d_v on edges (v, t^*). This is an $s^* - t^*$ flow in G' of value D.

If there is a flow in G' of value D, every edge out of s^* and into t^* must saturate its capacity, so we can delete those edges to get a circulation f in G with $f^{in}(v) - f^{out}(v) = d_v$ for all $v \in V$.

Since hte max flow in G' can be integer valued, there exists an integer valued circulation. □

22.1.1 Generalization 2 - Circulation with Demands and Lower Bounds

Suppose we also impose a constraint that the flow on edge e must be at least l_e. New capacity condition: $\forall e \in E, l_e \leq f_e \leq c_e$.

Start from an initial "circulation" with $f_o(e) = l_e$. This flow satisfies the capacity conditions, but not the demand conditions:

$$f_0^{in}(v) - f_0^{out}(v) = \sum_{e \text{ into } v} l_e - \sum_{e \text{ out of } v} l_e \equiv L_v$$

We want to add a flow f with

$$f_1^{in}(v) - f_1^{out}(v) = d_v - L_v$$

Using the remaining capacities $c_e - l_e$ on edge e.

So we construct a circulation with demands problem on the same graph with new capacities $c_e - l_e$ and demands $d_v - L_v$.

Theorem 22.2
There is a feasible circulation in the demands and lower boundsproblem iff there is a feasible solution in the corresponding circulation with demands problem. If all demands, capacities, and lower bounds are integers, and the problem is feasible, then there is an integer-valued circulation.

Proof. Suppose there is a circulation in the problem without lower bounds. This satisfies the capacity conditions, and

$$f^{in}(v) - f^{out}(v) = \sum_{e \text{ into } v} (f'(e) + l_e) - \sum_{e \text{ out of } v} (f'(e) + l_e) = L_v + d_v - L_v = d_v$$

so this satisfies the demand conditions.

Conversely, suppose there is a circulation f in the original problem. Let $f'(e) = f(e) - l(e)$. Then f' satisfies the capacity conditions, and

$$f'^{in} - f'^{out}(v) = \sum_{e \text{ into } v} (f(e) - l_e) - \sum_{e \text{ out of } v} (f(e) - l_e) = d_v$$

□