21 Bipartite Matching, Extensions to Max Flow

- Bipartite Matching
- Extension to the max flow problem
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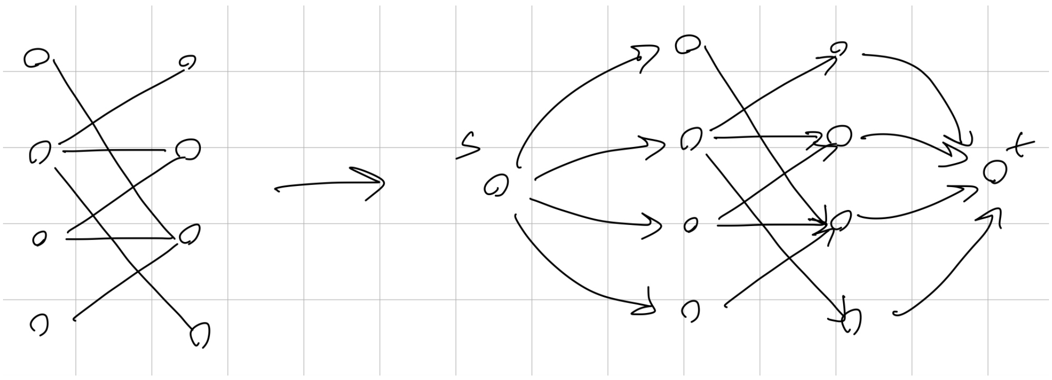
21.1 Bipartite Matching



We modify a bipartite graph into a flow network in the above way, with all edges having unit capacity.

> **Lemma 21.1**
> The size of a max matching in the given bipartite graph equals the value of a max flow in the corresponding flow network.

Proof. Given a matching M, construct a flow by sending 1 unit of flow along each corresponding edge of the flow network.

Also, send 1 unit of flow from s to every matched vertex in A, and 1 unit of flow from each matched vertex in B to t.

This is a valid flow of value $|M|$ (size of the matching).

Conversely, if there is a flow of value k, then there is an integer valued flow of value k.
Since there can be at most 1 unit of flow into each vertex A, each edge from A to B can have at most 1 unit of flow.

Considering the cut $(\{s\} \cup A,\ \{t\} \cup B)$, the flow leaving the cut is k, and there are k edges carrying flow from A to B.

Let M be the set of edges from A to B that carry flow.

To satisfy the conservation constraints, no two edges in M can share a vertex, so M is a matching. □

Running time: Supose $|A| = |B| = n$, $C = n$. The ford fulkerson running tiem is $O(n(m + n))$.

21.2 Extensions to the max flow problem

Generalization 0: having multiple sources and sinks.

Have multiple source nodes $s_1, s_2, \cdots, s_l$ and sink nodes $t_1, t_2, \cdots, t_l$. There is a true source node s that has edges from itself to every source node s_i with infinite capacity, and similarly a true sink node t that has edges of infinite capacity from the sink nodes t_i to itself.

Generalization 1: circulations with demands

Given a flow network $G = (V, E)$ with edge capacities.
Now each vertex has a **demand** d_v.

- $d_v > 0$ means v is a sink and should receive d_v more units of flow than it outputs.
- $d_v < 0$ means v is a source and should supply d_v more units of flow than it takes in.

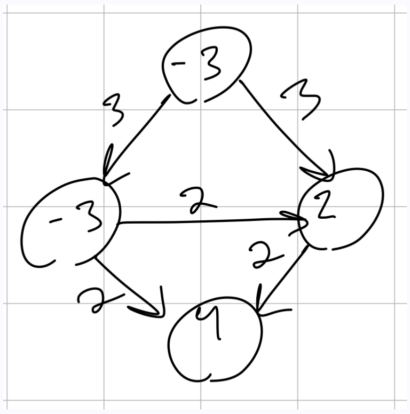
Let $S = \{v \in V : d_v < 0\}$ (sinks) and $T = \{v \in V : d_v > 0\}$ (sinks)

Definition 21.2
A **circulation** is a function $f : E \to \mathbb{R}^+$ satisfying

1. Capacity conditions: $\forall e \in E : 0 \leq f(e) \leq c_e$
2. Demand conditions: $\forall v \in V,\ f^{in}(v) - f^{out}(v) = d_v$

Here, the circulation problem is not an optimization problem, but a feasibility problem.

Example 21.3
Take the following flow network:



A valid flow for this network is the following: